An Initial Attempt of Solving DGLAP w/ A Spectral-Element Method

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Today's Goal

Solve

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2\pi}\alpha(t)\left[\int_x^1 \frac{1}{z}P\left(z\right)f\left(\frac{x}{z},t\right)\,\mathrm{d}z - \int_0^1 P\left(z\right)f\left(x,t\right)\,\mathrm{d}z\right], \text{ for } x \in \Omega \equiv [0,1]$$

where

$$\alpha(t) = \frac{12\pi}{\left(33 - 2n_F\right)t}$$

$$P(x) = \frac{4}{3}\frac{1+x^2}{1-x}$$

Ref: Olanj, N., Parsa, M. L., & Asgari, L. (2022). Analytical solution of the DGLAP equations using the generating function method. *Physics Letters B, 834*, 137472.

Today's Goal

Initial condition:

$$f(x,t=0)=f^0(x)=$$
$$1.3358x^{-0.6557}(1-x)^{2.2318}[1-0.26767(1-2\sqrt{x})-0.51620(2(1-2\sqrt{x})^2-1)$$
$$+0.47167(4(1-2\sqrt{x})^3-3(1-2\sqrt{x}))-0.12224(8(1-2\sqrt{x})^4-8(1-2\sqrt{x})^2$$
$$+1)]+3.6009x^{-0.7495}(1-x)^{4.6165}[1-1.3817(1-2\sqrt{x})+0.49690(2(1-2\sqrt{x})^2$$
$$-1)-0.040740(4(1-2\sqrt{x})^3-3((1-2\sqrt{x}))-0.03926(8((1-2\sqrt{x})^4-8((1-2\sqrt{x})^2+1)]$$
$$-0.53737x^{-0.08405}(1-x)^{14.402}(1-\frac{x}{0.056131})$$

Ref: Olanj, N., Parsa, M. L., & Asgari, L. (2022). Analytical solution of the DGLAP equations using the generating function method. *Physics Letters B, 834*, 137472.

Core Ideas of Spectral/Finite-Element Methods

- **Polynomial series w/ local supports**:

$$f(x) \approx \hat{f}(x) = \sum_{i=0}^{N_p} f_i \phi_i(x)$$

  The basis functions, $\phi_i(x)$, are non-zero only in local domains, i.e., elements.

- **Differentiation/integration approximations**:

$$\frac{\partial f}{\partial x} \not\approx \sum_{i=0}^{N_p} f_i \frac{\partial \phi_i(x)}{\partial x}$$

- **Minimizing multuple weighted residuals**:

$$\int_{\Omega} \psi_i(x) \mathcal{L}[\hat{f}(x)] \, \mathrm{d}x = 0$$

Warm-Up: Interpolation (1/5)

Approximation:

$$f(x) = \frac{35}{16} x^{-\frac{1}{2}} (1-x)^3 \approx \hat{f}(x) = \sum_{i=0}^{3} f_i \phi_i(x) \text{ for } x \in [0,1]$$

4 unknowns $\Rightarrow$ need 4 equations $\Rightarrow$ 4 weighted residuals:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \int_0^1 \psi_0(x) \left(\hat{f}(x) - f(x)\right) \, \mathrm{d}x \\ \int_0^1 \psi_1(x) \left(\hat{f}(x) - f(x)\right) \, \mathrm{d}x \\ \int_0^1 \psi_2(x) \left(\hat{f}(x) - f(x)\right) \, \mathrm{d}x \\ \int_0^1 \psi_3(x) \left(\hat{f}(x) - f(x)\right) \, \mathrm{d}x \end{bmatrix} = \begin{bmatrix} \int_0^1 \psi_0\phi_0 \, \mathrm{d}x & \cdots & \cdots & \int_0^1 \psi_0\phi_3 \, \mathrm{d}x \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \int_0^1 \psi_3\phi_0 \, \mathrm{d}x & \cdots & \cdots & \int_0^1 \psi_3\phi_3 \, \mathrm{d}x \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} - \begin{bmatrix} \int_0^1 \psi_0(x) f(x) \, \mathrm{d}x \\ \int_0^1 \psi_1(x) f(x) \, \mathrm{d}x \\ \int_0^1 \psi_2(x) f(x) \, \mathrm{d}x \\ \int_0^1 \psi_3(x) f(x) \, \mathrm{d}x \end{bmatrix}$$

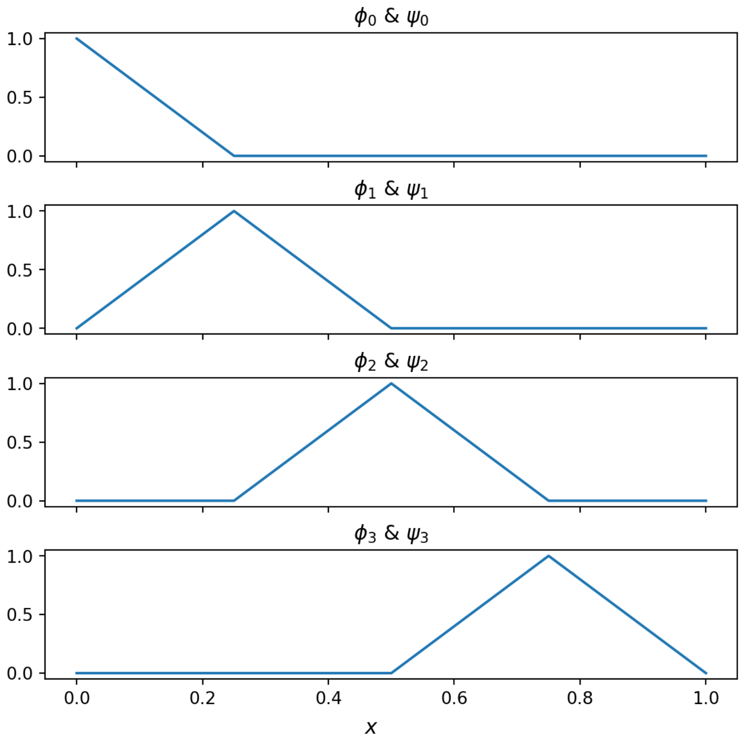
Warm-Up: Interpolation (2/5)

$$\phi_0(x) = \psi_0(x) = \begin{cases} 1 - 3x & \text{for } x \in [0, \frac{1}{3}] \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_1(x) = \psi_1(x) = \begin{cases} 3x & \text{for } x \in [0, \frac{1}{3}] \\ 2 - 3x & \text{for } x \in [\frac{1}{3}, \frac{2}{3}] \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \psi_2(x) = \begin{cases} -1 + 3x & \text{for } x \in [\frac{1}{3}, \frac{2}{3}] \\ 3 - 3x & \text{for } x \in [\frac{2}{3}, 1] \\ 0 & \text{otherwise} \end{cases}$$

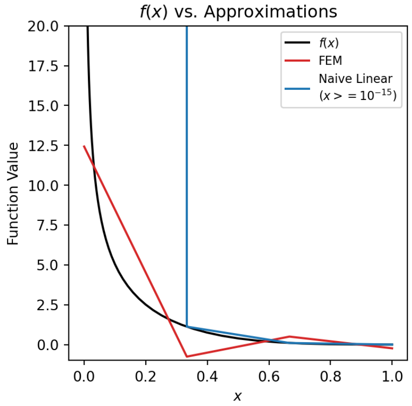
$$\phi_3(x) = \psi_3(x) = \begin{cases} -2 + 3x & \text{for } x \in [\frac{2}{3}, 1] \\ 0 & \text{otherwise} \end{cases}$$



Warm-Up: Interpolation (3/5)

Solving $\boldsymbol{A}\boldsymbol{f} - \boldsymbol{b} = \boldsymbol{0}$:

$$A \equiv \begin{bmatrix} \int_0^1 \psi_0\phi_0 \, \mathrm{d}x & \cdots & \cdots & \int_0^1 \psi_0\phi_3 \, \mathrm{d}x \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \int_0^1 \psi_3\phi_0 \, \mathrm{d}x & \cdots & \cdots & \int_0^1 \psi_3\phi_3 \, \mathrm{d}x \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & 0 & 0 \\ \frac{1}{18} & \frac{2}{9} & \frac{1}{18} & 0 \\ 0 & \frac{1}{18} & \frac{2}{9} & \frac{1}{18} \\ 0 & 0 & \frac{1}{18} & \frac{1}{9} \end{bmatrix}$$



$$\boldsymbol{b} \equiv \begin{bmatrix} \int_0^1 \psi_0(x) f(x) \, \mathrm{d}x \\ \vdots \\ \int_0^1 \psi_3(x) f(x) \, \mathrm{d}x \end{bmatrix} = \frac{1}{1458} \begin{bmatrix} 1172\sqrt{3} \\ 1985\sqrt{6} - 2344\sqrt{3} \\ 7776 - 3970\sqrt{6} + 1172\sqrt{3} \\ 1985\sqrt{6} - 4860 \end{bmatrix}, \; \hat{\boldsymbol{f}} \equiv \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \begin{bmatrix} 15.3\ldots \\ -0.42\ldots \\ 0.84\ldots \\ -0.11\ldots \end{bmatrix}$$

Warm-Up: Interpolation (4/5)

For an arbitrary N in $f(x) \approx \hat{f}(x) = \sum_{i=0}^{N} f_i \phi_i(x)$, $\Delta x = \frac{1}{N}$, and $x_i = i\Delta x$:

$$\phi_i(x) = \psi_i(x) = \begin{cases} \phi_{i,1}(x) = 1 - i + \frac{1}{\Delta x}x, & \text{if } x \in [x_{i-1}, x_i] \\ \phi_{i,2}(x) = 1 + i - \frac{1}{\Delta x}x, & \text{if } x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

$$\boldsymbol{A} = A_{i,j} = \begin{cases} 1 \,/\, 3N, & \text{if } i = j = 0 \text{ or } i = j = N \\ 2 \,/\, 3N, & \text{if } i = j \neq 0 \neq N \\ 1 \,/\, 6N, & \text{if } |i - j| = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\boldsymbol{b} = b_i \approx \begin{cases} \Delta x \sum_{k=1}^{N_q} \psi_i((\xi_k + i)\Delta x) f((\xi_k + i)\Delta x)\omega_k & \text{if } i \neq 0 \text{ and } i \neq N \\ \frac{\Delta x}{2} \sum_{k=1}^{N_q} \psi_i((\xi_k + 1)\frac{\Delta x}{2}) f((\xi_k + 1)\frac{\Delta x}{2})\omega_k & \text{if } i = 0 \\ \frac{\Delta x}{2} \sum_{k=1}^{N_q} \psi_i((\xi_k + 2N - 1)\frac{\Delta x}{2}) f((\xi_k + 2N - 1)\frac{\Delta x}{2})\omega_k & \text{if } i = N \end{cases}$$

Warm-Up: Interpolation (5/5)

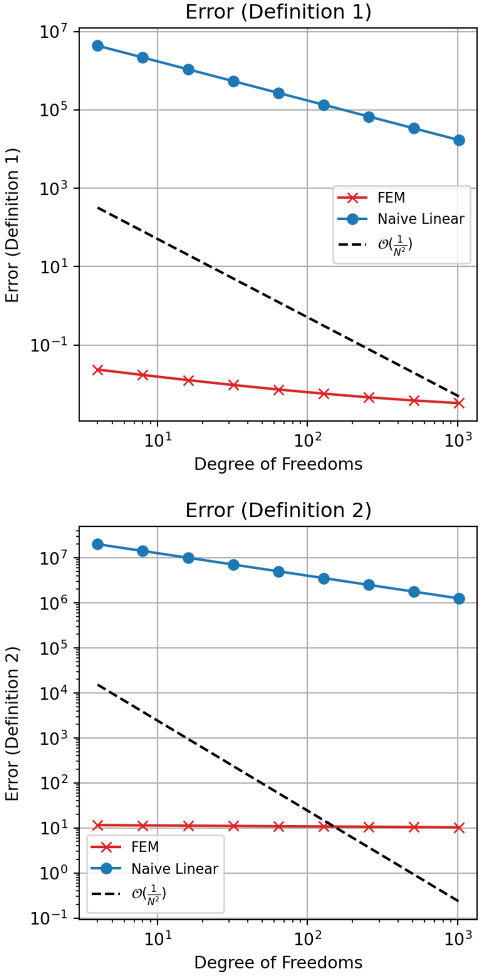
All about errors and their convergence:

- Error definition 1:

$$E_1 = \frac{\left| \int_{10^{-15}}^{1} f(x)\,\mathrm{d}x - \int_{10^{-15}}^{1} \hat{f}(x)\,\mathrm{d}x \right|}{\int_{10^{-15}}^{1} f(x)\,\mathrm{d}x}$$

- Error definition 2:

$$E_2 = \sqrt{\int_{10^{-15}}^{1} \left(f(x) - \hat{f}(x) \right)^2 \,\mathrm{d}x}$$



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After the Warm-Up, What's Next?

- Viewing from ***elements***: divide-and-conquer
  - Enabling non-uniform grid, non-conformal grid, h-refinement
- Solving the target integro-differential equation
- ***spectral***: better† than linear
  - Enabling p-refinement
- ***elements*** + ***spectral***
  - modern FEM, a.k.a. h/p FEM or spectral element methods

Linear Finite-Element in 1D (1/7)

$$f(x) \approx \hat{f}(x) = \sum_{i=1}^{N_e} \sum_{p=0}^{N_p} \hat{f}_p^i \phi_p^i(x) = \begin{cases} \sum_{p=0}^{N_p} \hat{f}_p^1 \phi_p^1(x) & \text{if } x \in \Omega^1 \\ \sum_{p=0}^{N_p} \hat{f}_p^2 \phi_p^2(x) & \text{if } x \in \Omega^2 \\ \vdots & \vdots \\ \sum_{p=0}^{N_p} \hat{f}_p^{N_e} \phi_i^{N_e}(x) & \text{if } x \in \Omega^{N_e} \end{cases}$$

- Superscripts: the indices of element
- Subscripts: quantities/functions associated with *nodes*
- For linear approximation, $N_p = 1$.
- $\hat{f}_{N_p}^i$ is not necessarily the same as $\hat{f}_0^{i+1}$ even if $x_{N_p}^i = x_0^{i+1}$.

Linear Finite-Element in 1D (2/7)

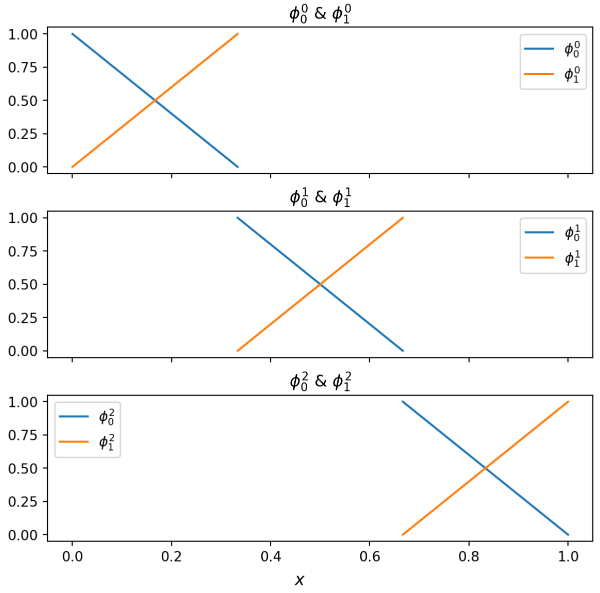
Standard element, $\Omega^{ST} \equiv [-1, 1]$:

$$\phi_0(\xi) = \begin{cases} \frac{1}{2}(1-\xi), & \text{if } \xi \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_1(\xi) = \begin{cases} \frac{1}{2}(1+\xi), & \text{if } \xi \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

Mapping to Ω^i for $i = 1, \cdots, N_e$:

$$\begin{cases} h^i(\xi) = \frac{\Delta x_i}{2}(\xi + 1) + x_0^i \\ g^i(x) = \frac{2}{\Delta x_i}(x - x_0^i) - 1 \end{cases} \Rightarrow \begin{cases} \phi_0^i(x) = \phi_0(g^i(x)) \\ \phi_1^i(x) = \phi_1(g^i(x)) \end{cases}$$



Linear Finite-Element in 1D (3/7)

Now $N_e(N_p+1)=6$ unknowns $\Rightarrow$ need 6 equations. The easiest way:

$$\psi_p^i(x)=\phi_p^i(x)$$

for $p=0,1$ and $i=1,2,3$.

Let's take a look at the two equations for the i-th element:

$$\int_{x\in\Omega^i}\psi_0^i(x)\hat{f}(x)\,\mathrm{d}x=\int_{x\in\Omega^i}\psi_0^i(x)f(x)\,\mathrm{d}x$$
$$\int_{x\in\Omega^i}\psi_1^i(x)\hat{f}(x)\,\mathrm{d}x=\int_{x\in\Omega^i}\psi_1^i(x)f(x)\,\mathrm{d}x$$

$$\Rightarrow\frac{\Delta x^i}{2}\begin{bmatrix}\int_{-1}^{1}\psi_0(\xi)\phi_0(\xi)\,\mathrm{d}\xi & \int_{-1}^{1}\psi_0(\xi)\phi_1(\xi)\,\mathrm{d}\xi\\ \int_{-1}^{1}\psi_1(\xi)\phi_0(\xi)\,\mathrm{d}\xi & \int_{-1}^{1}\psi_1(\xi)\phi_1(\xi)\,\mathrm{d}\xi\end{bmatrix}\begin{bmatrix}\hat{f}_0^i\\ \hat{f}_1^i\end{bmatrix}=\begin{bmatrix}\frac{\Delta x_i}{2}\int_{-1}^{1}\psi_0(\xi)f(h^i(\xi))\,\mathrm{d}\xi\\ \frac{\Delta x_i}{2}\int_{-1}^{1}\psi_1(\xi)f(h^i(\xi))\,\mathrm{d}\xi\end{bmatrix}$$

Linear Finite-Element in 1D (4/7)

Define

$$\Rightarrow \boldsymbol{A}^{ST} = A_{i,j}^{ST} \equiv \int_{-1}^{1} \psi_i \phi_j \,\mathrm{d}\xi = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The matrix for all equations from all elements:

$$\begin{bmatrix} \frac{\Delta x^1}{2} A_{0,0}^{ST} & \frac{\Delta x^1}{2} A_{0,1}^{ST} & 0 & 0 & 0 & 0 \\ \frac{\Delta x^1}{2} A_{1,0}^{ST} & \frac{\Delta x^1}{2} A_{1,1}^{ST} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Delta x^2}{2} A_{0,0}^{ST} & \frac{\Delta x^2}{2} A_{0,1}^{ST} & 0 & 0 \\ 0 & 0 & \frac{\Delta x^2}{2} A_{1,0}^{ST} & \frac{\Delta x^2}{2} A_{1,1}^{ST} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Delta x^3}{2} A_{0,0}^{ST} & \frac{\Delta x^3}{2} A_{0,1}^{ST} \\ 0 & 0 & 0 & 0 & \frac{\Delta x^3}{2} A_{1,0}^{ST} & \frac{\Delta x^3}{2} A_{1,1}^{ST} \end{bmatrix} \begin{bmatrix} \hat{f}_0^1 \\ \hat{f}_1^1 \\ \hat{f}_0^2 \\ \hat{f}_1^2 \\ \hat{f}_0^3 \\ \hat{f}_1^3 \end{bmatrix} = \begin{bmatrix} b_0^1 \\ b_1^1 \\ b_0^2 \\ b_1^2 \\ b_0^3 \\ b_1^3 \end{bmatrix}$$

Linear Finite-Element in 1D (5/7)

To enforce continuity, let's define:

$$\boldsymbol{T} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ so that } \begin{bmatrix} \hat{f}_0^1 \\ \hat{f}_1^1 \\ \hat{f}_0^2 \\ \hat{f}_1^2 \\ \hat{f}_0^3 \\ \hat{f}_1^3 \end{bmatrix} = \hat{\boldsymbol{f}} = \boldsymbol{T} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \boldsymbol{T}\hat{\boldsymbol{f}}_g$$

Also, note that the effect of $\boldsymbol{T}^\mathsf{T}$:

$$\boldsymbol{b}_g \equiv \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_0^1 \\ b_1^1 + b_0^2 \\ b_1^2 + b_0^3 \\ b_1^3 \end{bmatrix} = \boldsymbol{T}^\mathsf{T} \begin{bmatrix} b_0^1 \\ b_1^1 \\ b_0^2 \\ b_1^2 \\ b_0^3 \\ b_1^3 \end{bmatrix} \equiv \boldsymbol{T}^\mathsf{T}\boldsymbol{b}$$

Linear Finite-Element in 1D (6/7)

$$\boldsymbol{T}^\mathsf{T} \cdot \begin{bmatrix} \frac{\Delta x^1}{2} A^{ST}_{0,0} & \frac{\Delta x^1}{2} A^{ST}_{0,1} & 0 & 0 & 0 & 0 \\ \frac{\Delta x^1}{2} A^{ST}_{1,0} & \frac{\Delta x^1}{2} A^{ST}_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Delta x^2}{2} A^{ST}_{0,0} & \frac{\Delta x^2}{2} A^{ST}_{0,1} & 0 & 0 \\ 0 & 0 & \frac{\Delta x^2}{2} A^{ST}_{1,0} & \frac{\Delta x^2}{2} A^{ST}_{1,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\Delta x^3}{2} A^{ST}_{0,0} & \frac{\Delta x^3}{2} A^{ST}_{0,1} \\ 0 & 0 & 0 & 0 & \frac{\Delta x^3}{2} A^{ST}_{1,0} & \frac{\Delta x^3}{2} A^{ST}_{1,1} \end{bmatrix} \cdot \boldsymbol{T} \cdot \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \boldsymbol{T}^\mathsf{T} \cdot \begin{bmatrix} b^1_0 \\ b^1_1 \\ b^2_0 \\ b^2_1 \\ b^3_0 \\ b^3_1 \end{bmatrix}$$

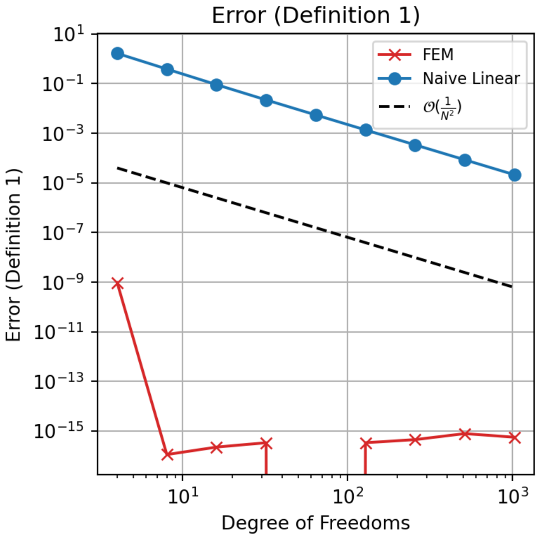
$$\Rightarrow \begin{bmatrix} \frac{\Delta x^1}{2} A^{ST}_{0,0} & \frac{\Delta x^1}{2} A^{ST}_{0,1} & 0 & 0 \\ \frac{\Delta x^1}{2} A^{ST}_{1,0} & \frac{\Delta x^1}{2} A^{ST}_{1,1} + \frac{\Delta x^2}{2} A^{ST}_{0,0} & \frac{\Delta x^2}{2} A^{ST}_{0,1} & 0 \\ 0 & \frac{\Delta x^2}{2} A^{ST}_{1,0} & \frac{\Delta x^2}{2} A^{ST}_{1,1} + \frac{\Delta x^3}{2} A^{ST}_{0,0} & \frac{\Delta x^3}{2} A^{ST}_{0,1} \\ 0 & 0 & \frac{\Delta x^3}{2} A^{ST}_{1,0} & \frac{\Delta x^3}{2} A^{ST}_{1,1} \end{bmatrix} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \begin{bmatrix} b^1_0 \\ b^1_1 + b^2_0 \\ b^2_1 + b^3_0 \\ b^3_1 \end{bmatrix}$$

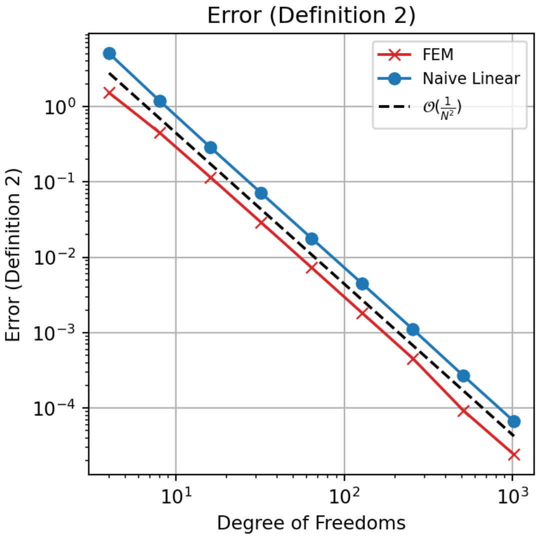
We can now solve the linear system to get $\hat{f}_i$ for $i = 0, \cdots, 3$.

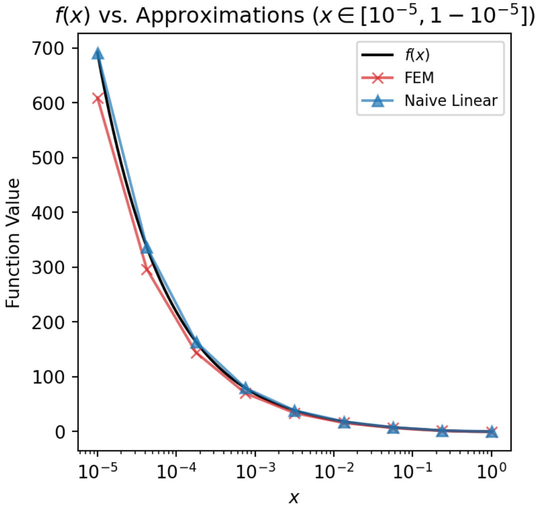
Linear Finite-Element in 1D (7/7)

Using log-uniform grid and a doamin of $[1e-5, 1-1e-5]$.

Error Definition 1	Error Definition 2
$E_1 = \frac{\left\lvert \int_{10^{-5}}^{1-10^{-5}} f(x)\,\mathrm{d}x - \int_{10^{-5}}^{1-10^{-5}} \hat{f}(x)\,\mathrm{d}x \right\rvert}{\int_{10^{-5}}^{1-10^{-5}} f(x)\,\mathrm{d}x}$	$E_2 = \sqrt{\int_{10^{-5}}^{1-10^{-5}} \left(f(x) - \hat{f}(x)\right)^2 \,\mathrm{d}x}$







Solving the Integro-Differential Equation (1/9)

Change of variable and aliases ($\tilde{\alpha}(t) \equiv \frac{1}{2\pi}\alpha(t)$ and $P_0 \equiv \int_0^1 P(z)\,\mathrm{d}z$):

$$\frac{\partial \hat{f}(x,t)}{\partial t} = \tilde{\alpha}(t)\left[\int_x^1 \frac{1}{z} P\left(\frac{x}{z}\right)\hat{f}\left(z,t\right)\,\mathrm{d}z - P_0 \hat{f}\left(x,t\right)\right]$$

Coefficients are functions of t:

$$f(x,t) \approx \sum_{i=1}^{N_e}\sum_{p=0}^{N_p} \hat{f}_p^i(t)\phi_p^i(x)$$

Note that we use $\sum_{i=1}^{N_e}$ because ϕ_p^i are preper defined zero region.

Solving the Integro-Differential Equation (2/9)

Weighted residual for the i-th element and p-th test function:

$$\int\limits_{x\in\Omega} \psi_p^i(x)\left\{\frac{\partial \hat{f}(x,t)}{\partial t} - \tilde{\alpha}(t)\left[\int\limits_x^1 \frac{1}{z}P\left(\frac{x}{z}\right)\hat{f}\left(z,t\right)\,\mathrm{d}z - P_0\hat{f}\left(x,t\right)\right]\right\}\mathrm{d}x = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int\limits_{x\in\Omega^i} \psi_p^i(x)\hat{f}(x,t)\,\mathrm{d}x - \tilde{\alpha}(t)\int\limits_{x\in\Omega^i} \psi_p^i(x)\left[\int\limits_x^1 \frac{1}{z}P\left(\frac{x}{z}\right)\hat{f}\left(z,t\right)\,\mathrm{d}z\right]\mathrm{d}x$$

$$+\tilde{\alpha}(t)P_0\int\limits_{x\in\Omega^i} \psi_p^i(x)\hat{f}\left(x,t\right)\,\mathrm{d}x = 0$$

Solving the Integro-Differential Equation (3/9)

The 1st and 3rd terms are the same as what we saw in the previous toy problem!

$$
\begin{cases}
\frac{\partial}{\partial t} \int\limits_{x\in\Omega^i} \psi_0^1 \hat{f}\,\mathrm{d}x - \tilde{\alpha} \int\limits_{x\in\Omega^i} \psi_0^1(x) \left[\int\limits_x^1 \frac{1}{z} P\left(\frac{x}{z}\right) \hat{f}(z,t)\,\mathrm{d}z \right] \mathrm{d}x + \tilde{\alpha} P_0 \int\limits_{x\in\Omega^i} \psi_0^1 \hat{f}\,\mathrm{d}x = 0 \\
\frac{\partial}{\partial t} \int\limits_{x\in\Omega^i} \psi_1^1 \hat{f}\,\mathrm{d}x - \tilde{\alpha} \int\limits_{x\in\Omega^i} \psi_1^1(x) \left[\int\limits_x^1 \frac{1}{z} P\left(\frac{x}{z}\right) \hat{f}(z,t)\,\mathrm{d}z \right] \mathrm{d}x + \tilde{\alpha} P_0 \int\limits_{x\in\Omega^i} \psi_1^1 \hat{f}\,\mathrm{d}x = 0 \\
\vdots \\
\vdots \\
\frac{\partial}{\partial t} \int\limits_{x\in\Omega^i} \psi_0^{N_e} \hat{f}\,\mathrm{d}x - \tilde{\alpha} \int\limits_{x\in\Omega^i} \psi_0^{N_e}(x) \left[\int\limits_x^1 \frac{1}{z} P\left(\frac{x}{z}\right) \hat{f}(z,t)\,\mathrm{d}z \right] \mathrm{d}x + \tilde{\alpha} P_0 \int\limits_{x\in\Omega^i} \psi_0^{N_e} \hat{f}\,\mathrm{d}x = 0 \\
\frac{\partial}{\partial t} \int\limits_{x\in\Omega^i} \psi_1^{N_e} \hat{f}\,\mathrm{d}x - \tilde{\alpha} \int\limits_{x\in\Omega^i} \psi_1^{N_e}(x) \left[\int\limits_x^1 \frac{1}{z} P\left(\frac{x}{z}\right) \hat{f}(z,t)\,\mathrm{d}z \right] \mathrm{d}x + \tilde{\alpha} P_0 \int\limits_{x\in\Omega^i} \psi_1^{N_e} \hat{f}\,\mathrm{d}x = 0
\end{cases}
$$

Solving the Integro-Differential Equation (4/9)

The 2nd term for the i-th element and p-th test function:

$$\int\limits_{x\in\Omega^i} \psi_p^i(x)\left[\int\limits_x^1 \frac{1}{z} P\left(\frac{x}{z}\right)\hat{f}\left(z,t\right)\,\mathrm{d}z\right]\mathrm{d}x = \int\limits_{x\in\Omega^i} \psi_p^i(x)\left[\sum_{k=1}^{N_e}\int\limits_{z\in\Omega^k} \frac{1}{z} P\left(\frac{x}{z}\right)\hat{f}\left(z,t\right)\,\mathrm{d}z\right]\mathrm{d}x$$

$$= \left[\iint\limits_{\substack{x\in\Omega^i\\ z\in\Omega^1}} \tfrac{1}{z}P\left(\tfrac{x}{z}\right)\psi_p^i(x)\phi_0^1(z)\,\mathrm{d}z\,\mathrm{d}x \quad \cdots \quad \iint\limits_{\substack{x\in\Omega^i\\ z\in\Omega^{N_e}}} \tfrac{1}{z}P\left(\tfrac{x}{z}\right)\psi_p^i(x)\phi_1^{N_e}(z)\,\mathrm{d}z\,\mathrm{d}x\right]\cdot\hat{\boldsymbol{f}}(t)$$

$$= \left[0 \quad \cdots \quad \iint\limits_{\substack{x\in\Omega^i\\ z\in\Omega^i}} \tfrac{1}{z}P\left(\tfrac{x}{z}\right)\psi_p^i(x)\phi_0^i(z)\,\mathrm{d}z\,\mathrm{d}x \quad \cdots \quad \iint\limits_{\substack{x\in\Omega^i\\ z\in\Omega^{N_e}}} \tfrac{1}{z}P\left(\tfrac{x}{z}\right)\psi_p^i(x)\phi_1^{N_e}(z)\,\mathrm{d}z\,\mathrm{d}x\right]\cdot\hat{\boldsymbol{f}}(t)$$

Note that the integrals are zeros because when $z < x_0^i$.

Solving the Integro-Differential Equation (5/9)

$$\boldsymbol{B}_{i,j} \equiv \begin{bmatrix} \iint\limits_{\substack{x\in\Omega^i \\ z\in\Omega^j}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_0^i(x)\phi_0^j(z)\,\mathrm{d}z\,\mathrm{d}x & \iint\limits_{\substack{x\in\Omega^i \\ z\in\Omega^j}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_0^i(x)\phi_1^j(z)\,\mathrm{d}z\,\mathrm{d}x \\ \iint\limits_{\substack{x\in\Omega^i \\ z\in\Omega^j}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_1^i(x)\phi_0^j(z)\,\mathrm{d}z\,\mathrm{d}x & \iint\limits_{\substack{x\in\Omega^i \\ z\in\Omega^j}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_1^i(x)\phi_1^j(z)\,\mathrm{d}z\,\mathrm{d}x \end{bmatrix}$$

$$\boldsymbol{B}_i^D \equiv \begin{bmatrix} \iint\limits_{\substack{x\in\Omega^i \\ z\in[x,x_1^i]}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_0^i(x)\phi_0^i(z)\,\mathrm{d}z\,\mathrm{d}x & \iint\limits_{\substack{x\in\Omega^i \\ z\in[x,x_1^i]}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_0^i(x)\phi_1^i(z)\,\mathrm{d}z\,\mathrm{d}x \\ \iint\limits_{\substack{x\in\Omega^i \\ z\in[x,x_1^i]}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_1^i(x)\phi_0^i(z)\,\mathrm{d}z\,\mathrm{d}x & \iint\limits_{\substack{x\in\Omega^i \\ z\in[x,x_1^i]}} \frac{1}{z} P\left(\frac{x}{z}\right) \psi_1^i(x)\phi_1^i(z)\,\mathrm{d}z\,\mathrm{d}x \end{bmatrix}$$

Note that $\boldsymbol{B}_i^D$ needs special quadrature rules for triangle domains to avoid the singularity on the diagonal of (x, z)-plane.

Solving the Integro-Differential Equation (6/9)

$$B = \begin{bmatrix} \boldsymbol{B}_1^D & \boldsymbol{B}_{1,2} & \boldsymbol{B}_{1,3} & \cdots & \boldsymbol{B}_{1,N_e} \\ \boldsymbol{0} & \boldsymbol{B}_2^D & \boldsymbol{B}_{2,3} & \cdots & \boldsymbol{B}_{2,N_e} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{B}_3^D & \cdots & \boldsymbol{B}_{3,N_e} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \cdots & \boldsymbol{0} & \boldsymbol{B}_{N_e}^D \end{bmatrix}$$

May need more work to make it memory efficient.

Solving the Integro-Differential Equation (7/9)

Put everything together:

$$\boldsymbol{A}\frac{\partial}{\partial t}\hat{\boldsymbol{f}}(t) - \tilde{\alpha}(t)\boldsymbol{B}\hat{\boldsymbol{f}}(t) + \tilde{\alpha}(t)P_0\boldsymbol{A}\hat{\boldsymbol{f}}(t) = 0$$

With a naive temporal discretization for now:

$$\boldsymbol{A}\frac{\hat{\boldsymbol{f}}^{n+1} - \hat{\boldsymbol{f}}^{n+1}}{\Delta t} - \tilde{\alpha}^n\boldsymbol{B}\hat{\boldsymbol{f}}^n + \tilde{\alpha}^n P_0\boldsymbol{A}\hat{\boldsymbol{f}}^n = 0$$

Or

$$\boldsymbol{A}\hat{\boldsymbol{f}}^{n+1} = \left[(1 - \Delta t\tilde{\alpha}^n P_0)\boldsymbol{A} + \Delta t\tilde{\alpha}^n\boldsymbol{B}\right]\hat{\boldsymbol{f}}^n$$

Solving the Integro-Differential Equation (8/9)

Apply the global-local transform matrices $\boldsymbol{T}$ and $\boldsymbol{T}^{\mathsf{T}}$:

$$\left(\boldsymbol{T}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{T}\right)\hat{\boldsymbol{f}}_g^{n+1} = \left[(1-\Delta t\tilde{\alpha}^n P_0)\left(\boldsymbol{T}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{T}\right) + \Delta t\tilde{\alpha}^n\left(\boldsymbol{T}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{T}\right)\right]\hat{\boldsymbol{f}}^n$$

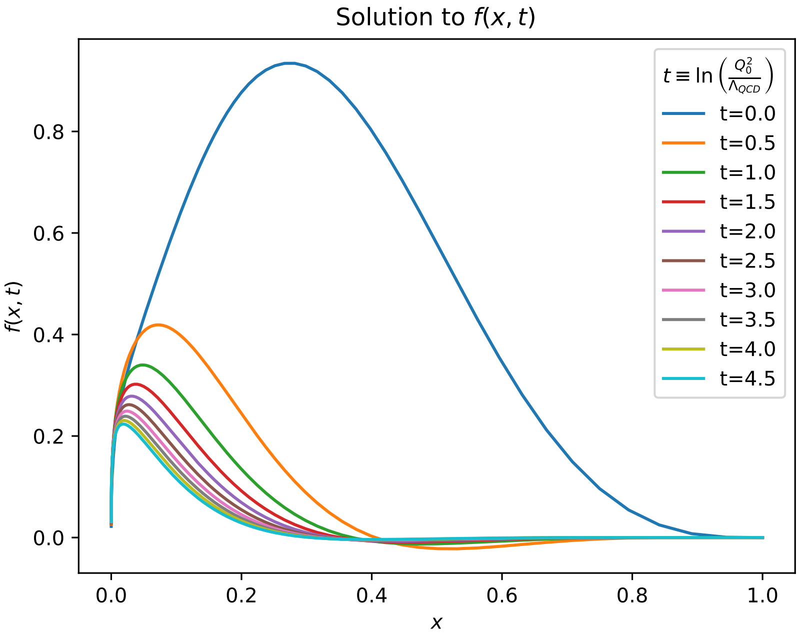
$$\Rightarrow \boldsymbol{A}_g\hat{\boldsymbol{f}}_g^{n+1} = \left[(1-\Delta t\tilde{\alpha}^n P_0)\boldsymbol{A}_g + \Delta t\tilde{\alpha}^n\boldsymbol{B}_g\right]\hat{\boldsymbol{f}}^n$$

If not dealing with million+ of unknowns, just pre-calculate the inverse matrix:

$$\hat{\mathbf{f}}_g^0 = \mathbf{A}_g^{-1}\mathbf{T}^{\mathsf{T}}\mathbf{b}$$

$$\hat{\mathbf{f}}_g^{n+1} = \left[(1-\Delta t\tilde{\alpha}^n P_0)\mathbf{I} + \Delta t\tilde{\alpha}^n\mathbf{A}_g^{-1}\mathbf{B}_g\right]\hat{\mathbf{f}}^n$$

Solving the Integro-Differential Equation (9/9)



Caveats:

- No solution for verification
- No convergence study
- Some parameters by guessing:
  - $n_F = 3$
  - $t_0 = \ln\left(\frac{Q_0^2}{\Lambda_{QCD}}\right) = 0$
- Improper integration for $\int_0^1 P(z)\,\mathrm{d}z$

Near-Future Directions

- Vector function of QCFs
- Projecting everything to the space of basis functions!!
- Higher order basis functions, ϕ_p^i
  - Types of polynomials matter, critical to numerical properties, e.g., condition numbers, matrix sparsities, etc.
- Custom test functions, ψ_p^i

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