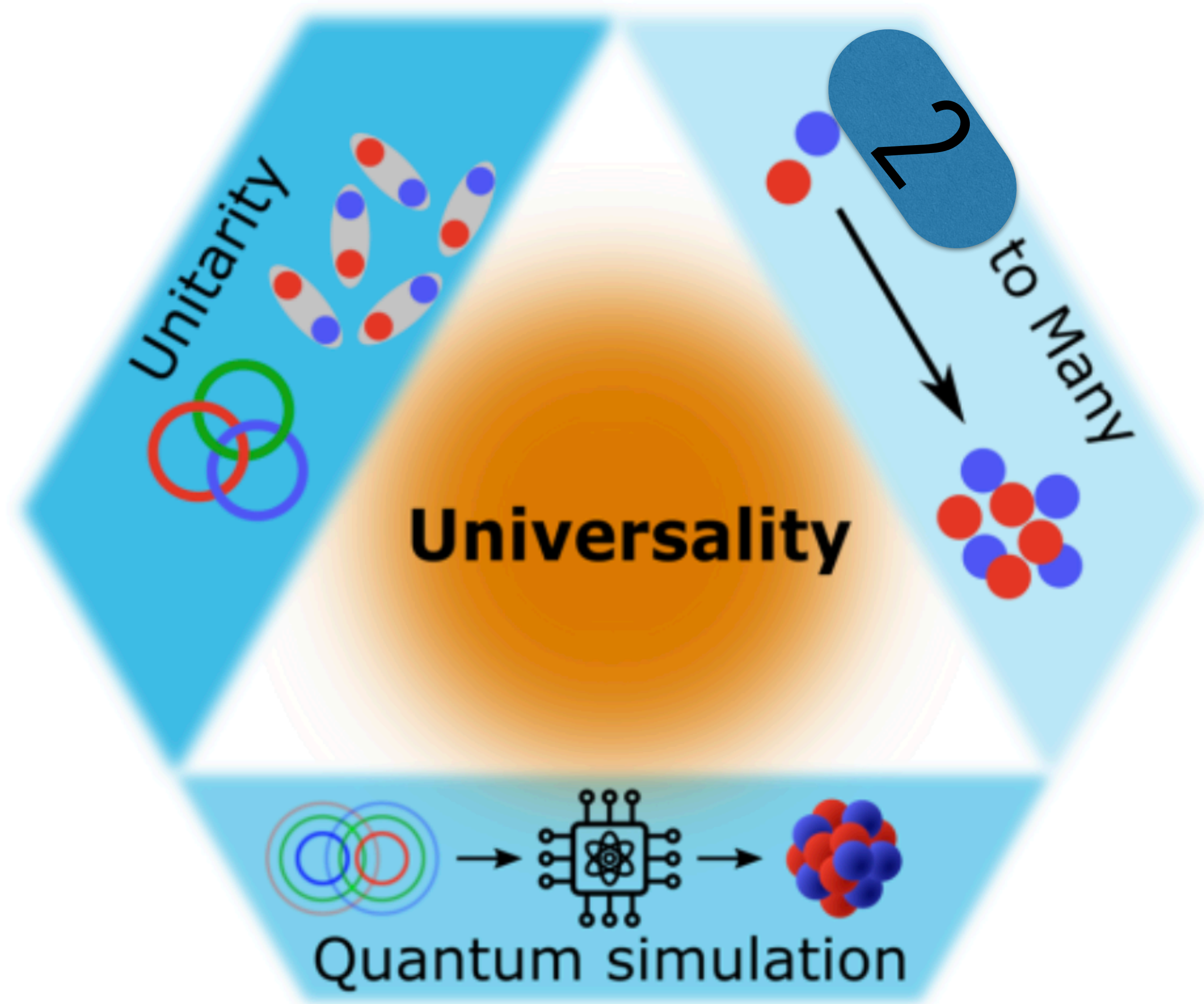
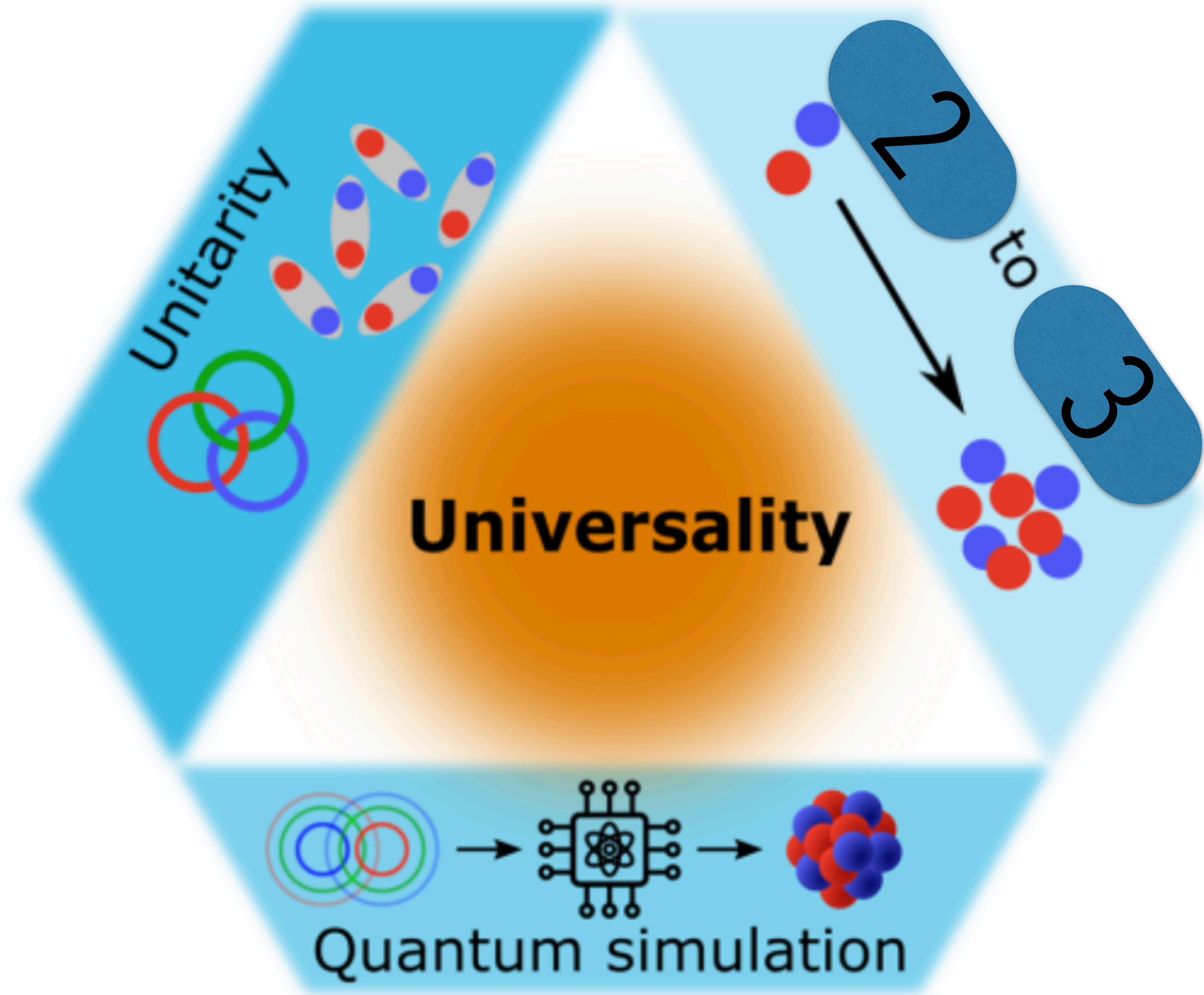


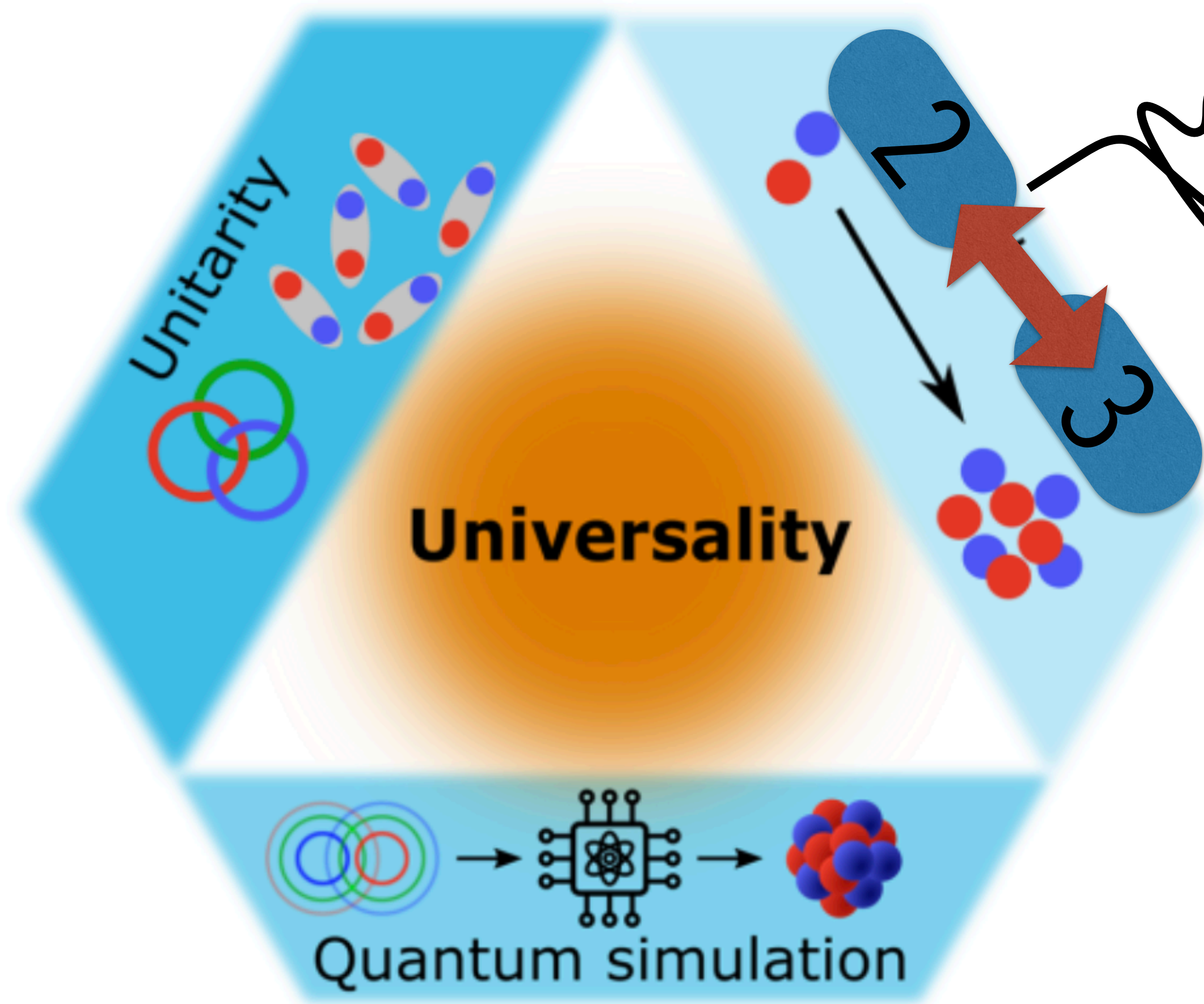
Caroline S. R. Costa



Caroline S. R. Costa



Caroline S. R. Costa



Hadrons systems

Caroline S. R. Costa

- ❑ 3N forces play a crucial role in describing nuclear properties
- ❑ Even at the 2-body sector, there are tensions → 2n scattering length
- ❑ Traditionally, nuclear properties and nuclei formation have been described using phenomenological models and / or EFTs

A New Class of Three Nucleon Forces and their Implications

Vincenzo Cirigliano,^{1,*} Maria Dawid,^{1,†} Wouter Dekens,^{1,‡} and Sanjay Reddy^{1,§}

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(Dated: November 4, 2024)

We identify a new class of three-nucleon forces that arises in the low-energy effective theory of nuclear interactions including pions. We estimate their contribution to the energy of neutron and nuclear matter and find that it can be as important as the leading-order three-nucleon forces previously considered in the literature. The magnitude of this force is set by the strength of the coupling of pions to two nucleons and is presently not well constrained by experiments. The implications for nuclei, nuclear matter, and the equation of state of neutron matter are briefly discussed.

- ❑ Motivation: Constrain two- and three-body forces directly from Standard Model



Crucial also for probing
BSM physics, e.g., in $0\nu\beta\beta$

outline

- relativistic integral equations [4D \rightarrow 3D]
- angular momentum projection of OPE
- angular momentum projection of amplitudes
- LSZ for 3Body \rightarrow 2Body
- toy models for 3pi [including isospin in the OPE]
- Numerical solutions // Unitary check

Hansen, RB, Edwards, Thomas, & Wilson (2020)

Jackura, RB, Dawid, Islam, & McCarty (2020)

Dawid, Islam, & RB (2023)

Jackura, RB (2023)

RB, S. R. Costa, Jackura, (2024)

Dawid, RB, Islam, Jackura, (2023)

Two-hadron systems

- Sum over all $2 \rightarrow 2$ amputated diagrams

$$i\mathcal{M}_2 = \text{[solid black vertex]} = \text{[white vertex]} + \text{[white vertex with one loop]} + \text{[white vertex with two loops]} + \dots$$

Two-hadron systems

□ Sum over all $2 \rightarrow 2$ amputated diagrams

$$i\mathcal{M}_2 = \text{[black blob]} = \text{[tree]} + \text{[1-loop]} + \text{[2-loop]} + \dots$$

$\left\{ \text{[orange tree]} + \text{[orange 1-loop]} + \dots + \text{[orange 2-loop]} + \dots \right\}$

All 2-PI s-channel diagrams

Two-hadron systems

Sum over all $2 \rightarrow 2$ amputated diagrams

$$i\mathcal{M}_2 = \text{blob} = \text{tree} + \text{1-loop} + \text{2-loop} + \dots$$

All 2-PI s-channel diagrams

The goal: Isolate all the singularities of the scattering amplitude!

Two-hadron systems

Sum over all $2 \rightarrow 2$ amputated diagrams

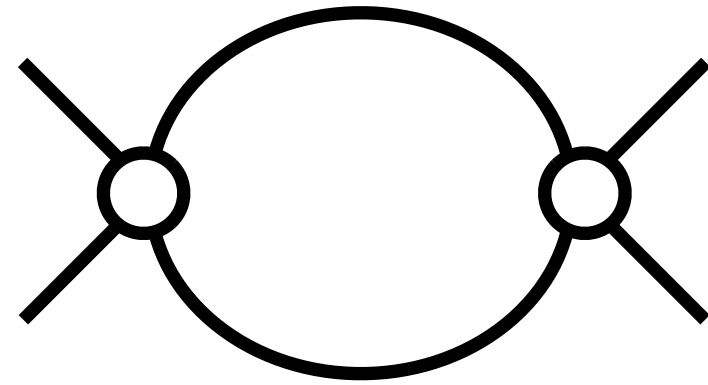
$$i\mathcal{M}_2 = \text{[black blob]} = \text{[tree]} + \text{[1-loop]} + \text{[2-loops]} + \dots$$

All 2-PI s-channel diagrams

- The goal: Isolate all the singularities of the scattering amplitude!
- Kernel is not singular in the kinematic region of interest
- Singularities are due to intermediate particles going on-shell

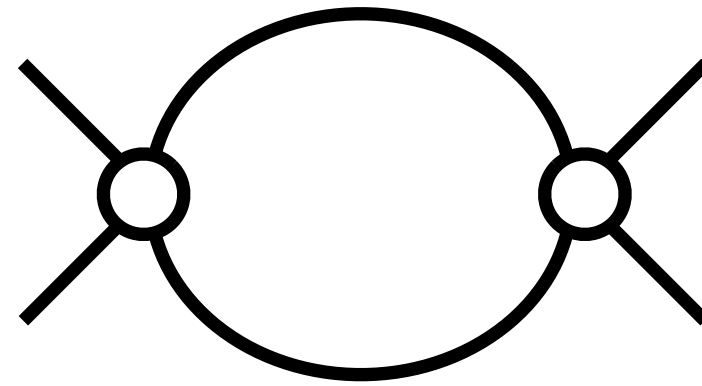
Two-hadron systems

- The goal: Isolate singularities from two particle on shell states in bubble diagrams



Two-hadron systems

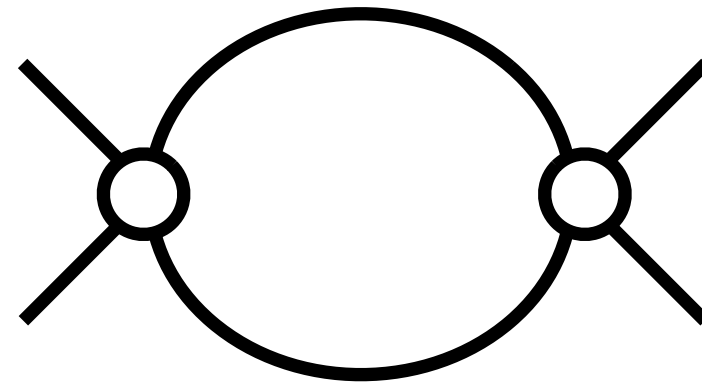
- The goal: Isolate singularities from two particle on shell states in bubble diagrams



$$= \int \frac{d^4 k}{(2\pi)^4} [iB(k, P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P - k)^2 - m^2 + i\epsilon}$$

Two-hadron systems

- The goal: Isolate singularities from two particle on shell states in bubble diagrams

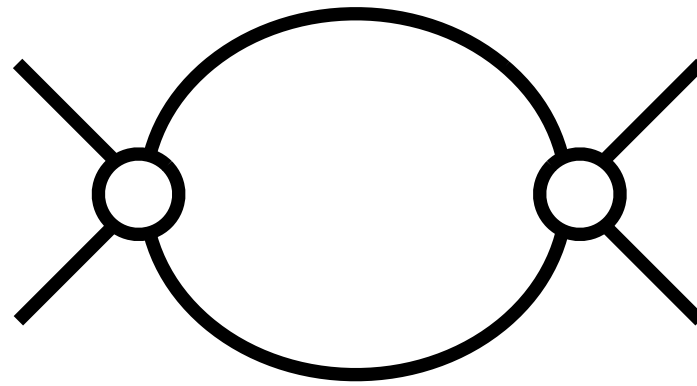


$$= \int \frac{d^4 k}{(2\pi)^4} [iB(k, P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P - k)^2 - m^2 + i\epsilon}$$

$$= \int \frac{dk_0}{2\pi} \frac{d^3 \vec{k}}{(2\pi)^3} [iB(k, P)]^2 \frac{i}{k_0^2 - \omega_k^2 + i\epsilon} \frac{i}{[(k_0 - E)^2 - \omega_{kp}^2 + i\epsilon]}$$

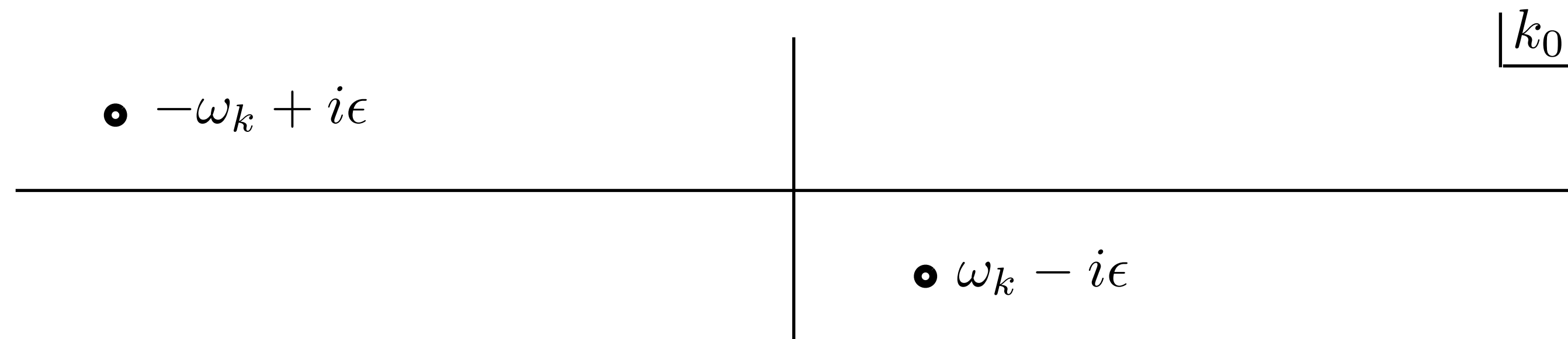
Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams



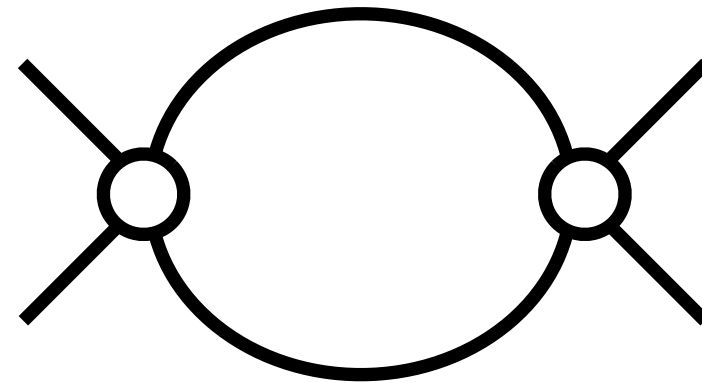
$$= \int \frac{d^4 k}{(2\pi)^4} [iB(k, P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P - k)^2 - m^2 + i\epsilon}$$

$$= \int \frac{dk_0}{2\pi} \frac{d^3 \vec{k}}{(2\pi)^3} [iB(k, P)]^2 \frac{i}{k_0^2 - \omega_k^2 + i\epsilon} \frac{i}{[(k_0 - E)^2 - \omega_{kp}^2 + i\epsilon]}$$



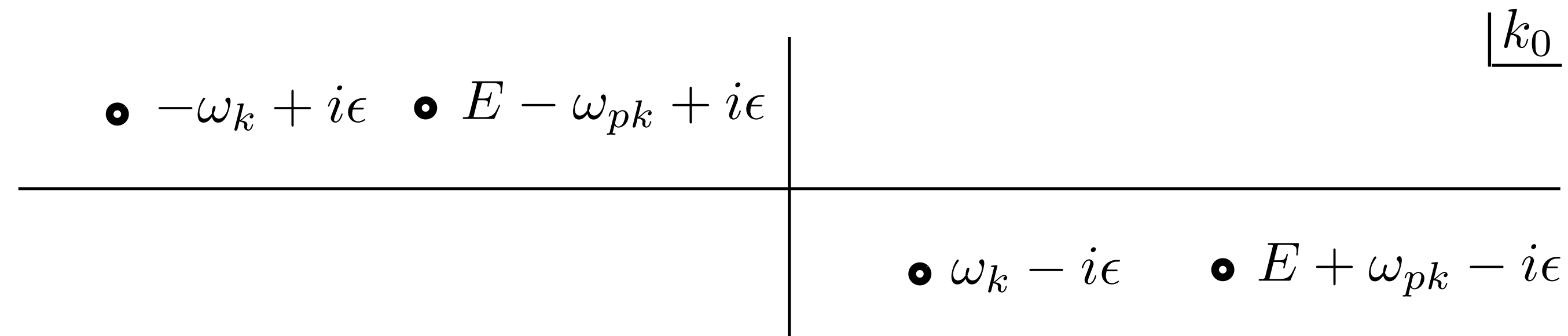
Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams



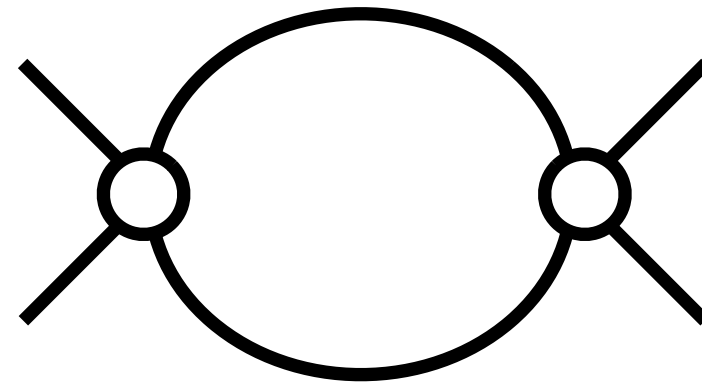
$$= \int \frac{d^4 k}{(2\pi)^4} [iB(k, P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P - k)^2 - m^2 + i\epsilon}$$

$$= \int \frac{dk_0}{2\pi} \frac{d^3 \vec{k}}{(2\pi)^3} [iB(k, P)]^2 \frac{i}{k_0^2 - \omega_k^2 + i\epsilon} \frac{i}{[(k_0 - E)^2 - \omega_{pk}^2 + i\epsilon]}$$



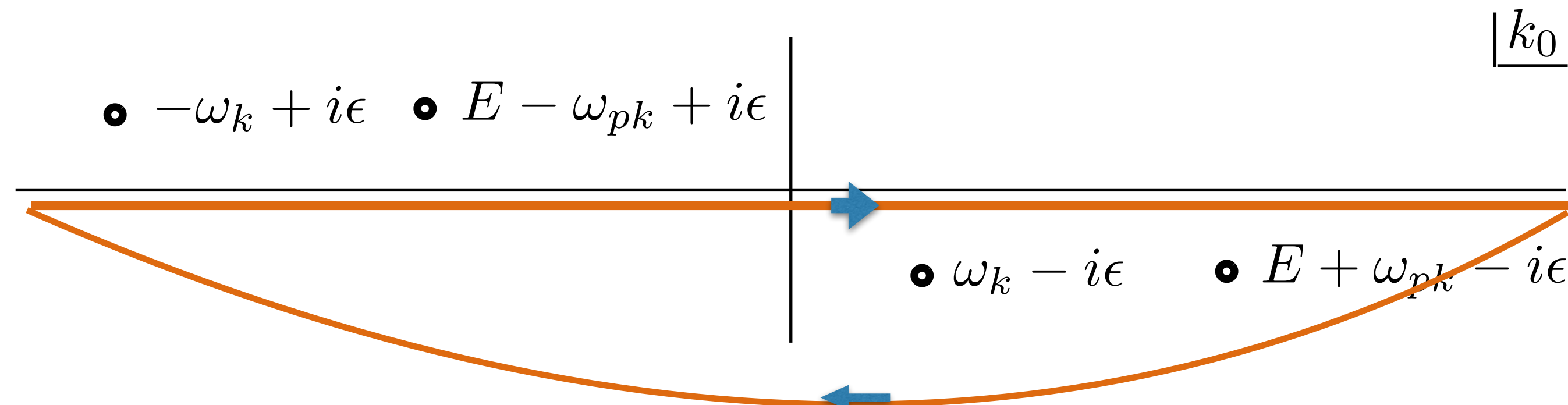
Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams



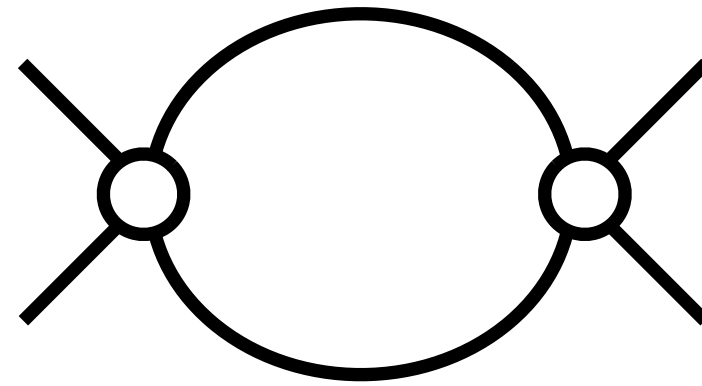
$$= \int \frac{d^4 k}{(2\pi)^4} [iB(k, P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P - k)^2 - m^2 + i\epsilon}$$

$$= \int \frac{dk_0}{2\pi} \frac{d^3 \vec{k}}{(2\pi)^3} [iB(k, P)]^2 \frac{i}{k_0^2 - \omega_k^2 + i\epsilon} \frac{i}{[(k_0 - E)^2 - \omega_{pk}^2 + i\epsilon]}$$



Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams

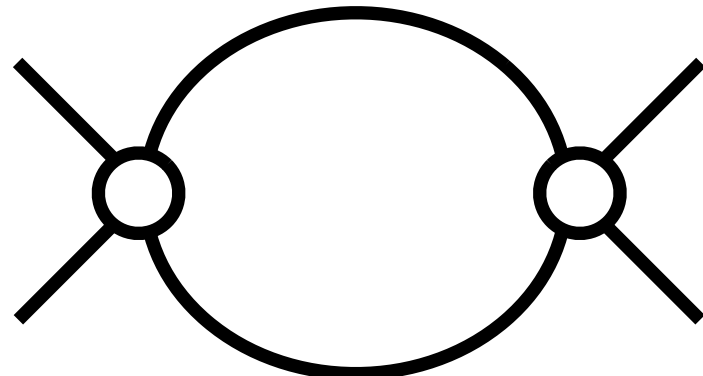


$$= \text{bubble diagram with dashed line} + \text{“smooth”}$$

The diagram shows an equals sign followed by a bubble diagram with a vertical dashed line connecting the top of the two small circles. This is followed by a plus sign and the word "smooth" in quotes.

Two-hadron systems

Isolate singularities from two particle on shell states in bubble diagrams

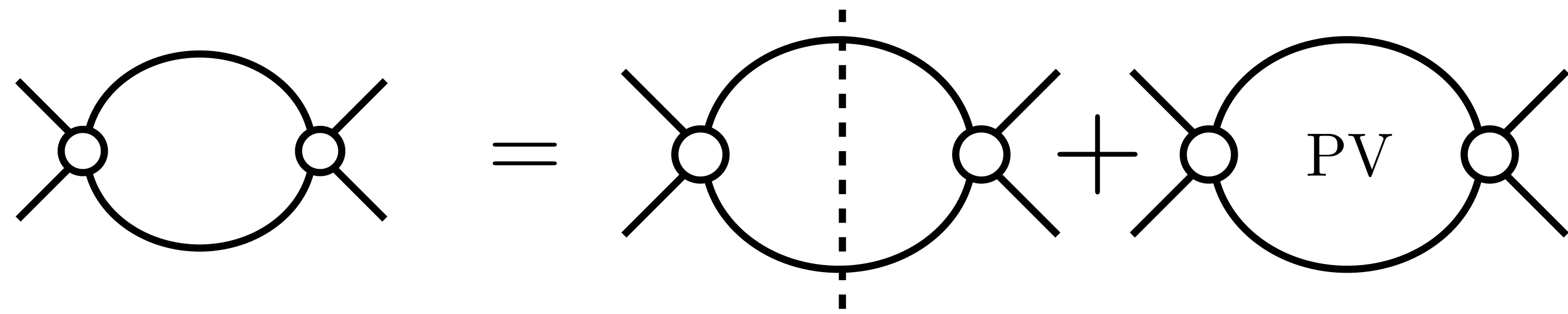


$$= \text{Bubble diagram with a vertical dashed line through the top} + \text{“smooth”}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{[iB(k, P)]^2}{(2\omega_k)^2} \pi \delta(E - 2\omega_k) + \text{“PV integral”}$$

$$= [iB_{on}] \rho [iB_{on}] + \text{“PV integral”}$$

$$\rho \equiv \frac{p}{8\pi E} \sim \sqrt{s - s_{th}} \quad \text{“Square root singularity”}$$



$$\text{Diagram} = \text{Diagram with dashed line} + \text{Diagram with PV}$$

The diagram shows a circle with two external lines on the left and two on the right. This is equal to the sum of two diagrams: one with a vertical dashed line through the circle, and another with the letters "PV" inside the circle.



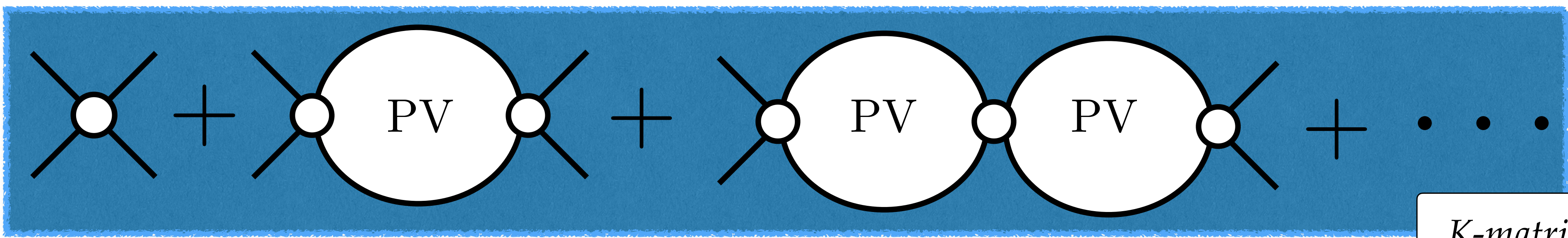
$$i\mathcal{M}_2 = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The equation shows a series of diagrams: a vertex with four external lines, a circle with two external lines, a diagram with two circles connected at a central vertex, and an ellipsis indicating further terms.

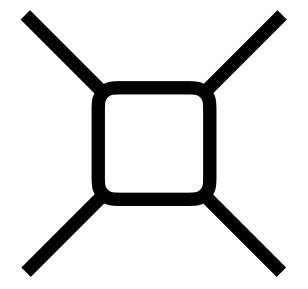
$$i\mathcal{M}_2 = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

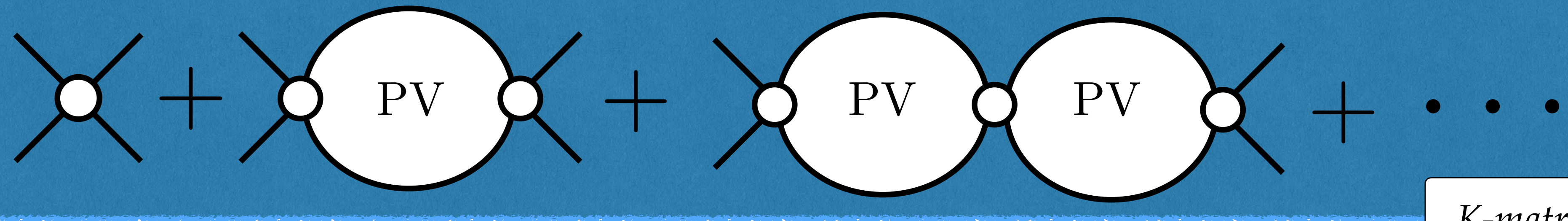
The image shows a series of Feynman diagrams representing the second-order scattering amplitude $i\mathcal{M}_2$. The first diagram is a four-point contact interaction. The second diagram is a bubble diagram with two external lines on each side and a central circle labeled "PV". The third diagram is a chain of two such bubbles. The series continues with an ellipsis.

$i\mathcal{M}_2 =$

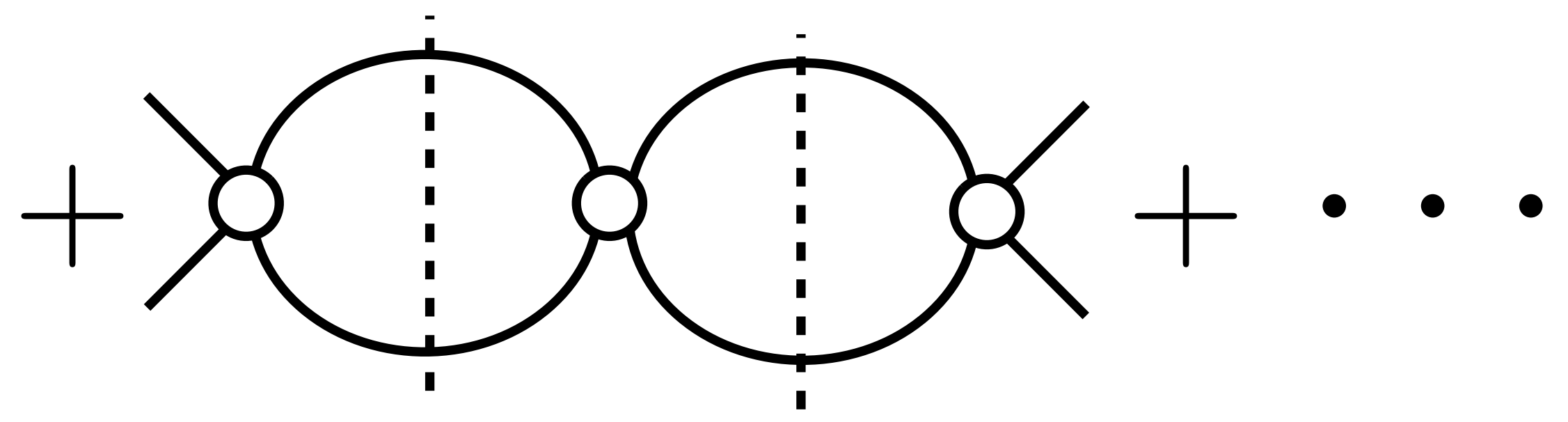
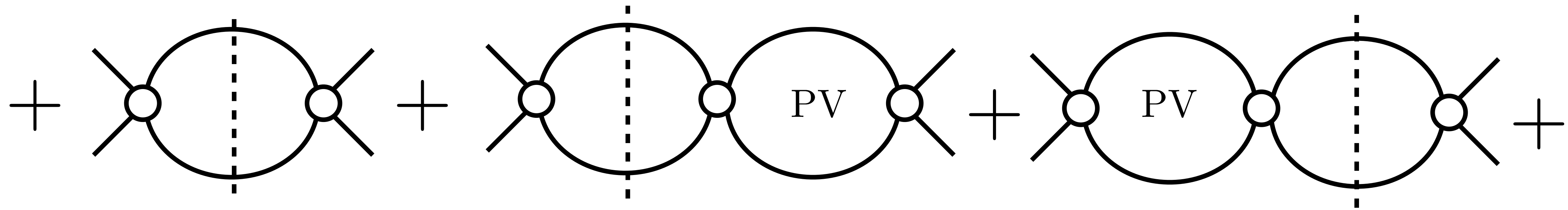
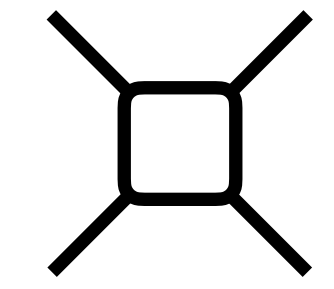


K-matrix

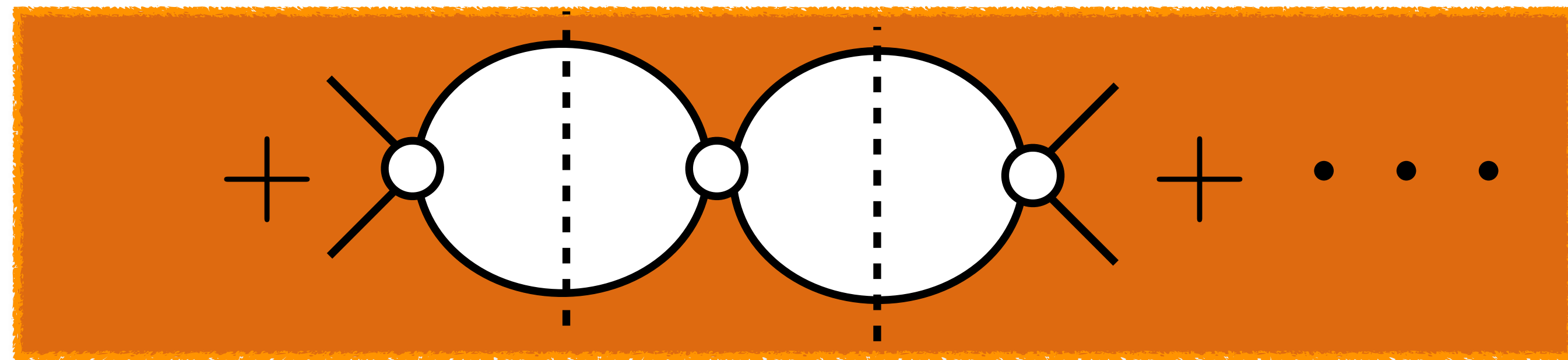
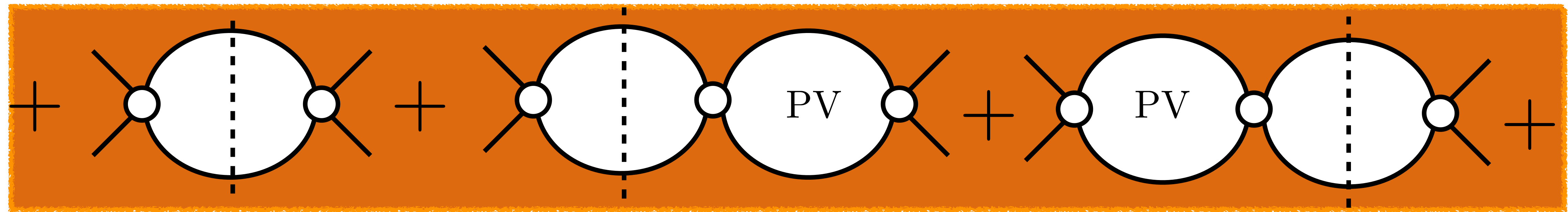
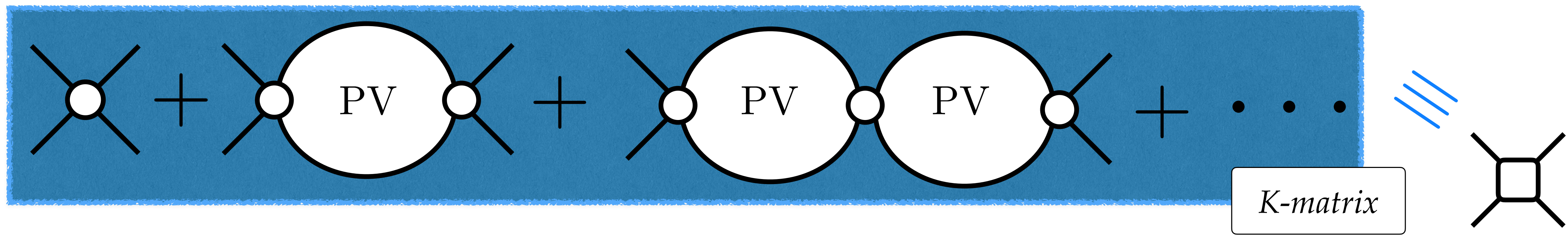


$i\mathcal{M}_2 =$ 

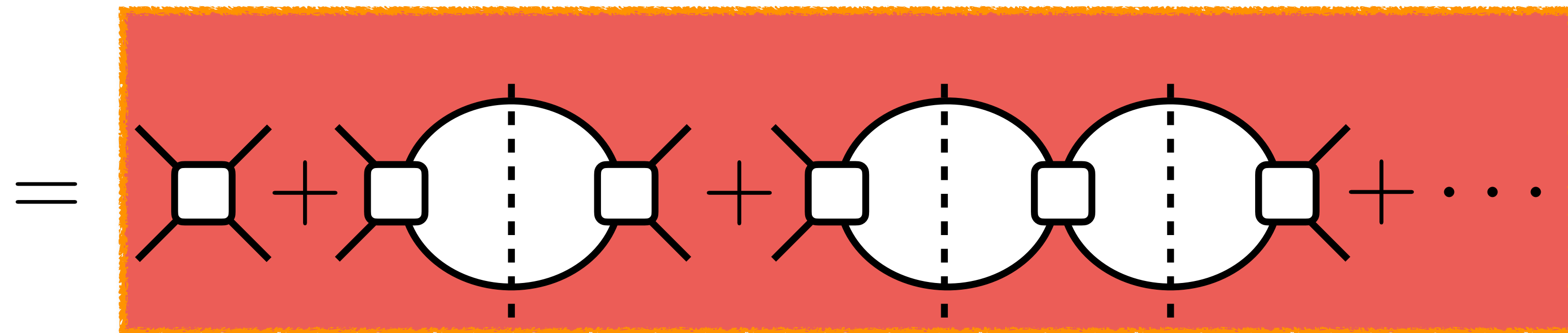
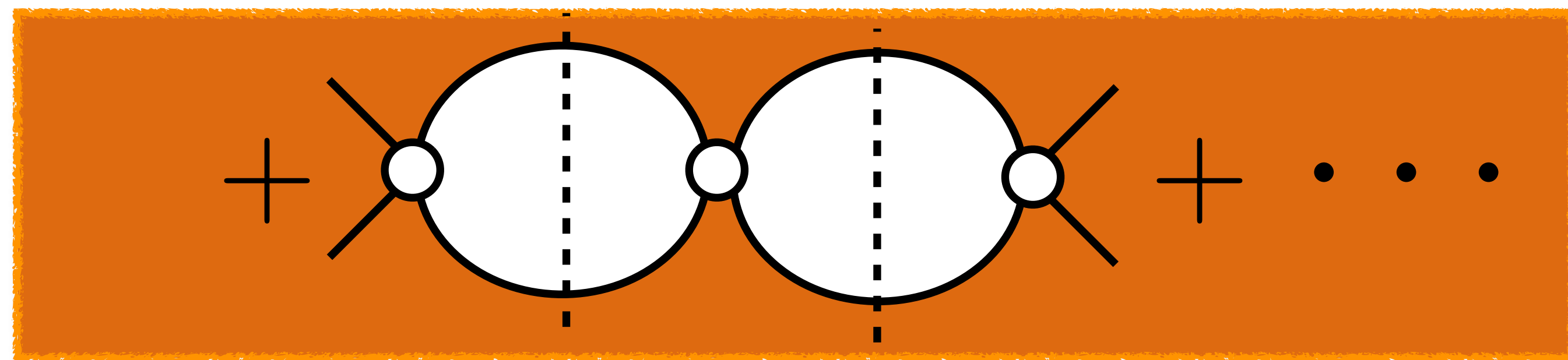
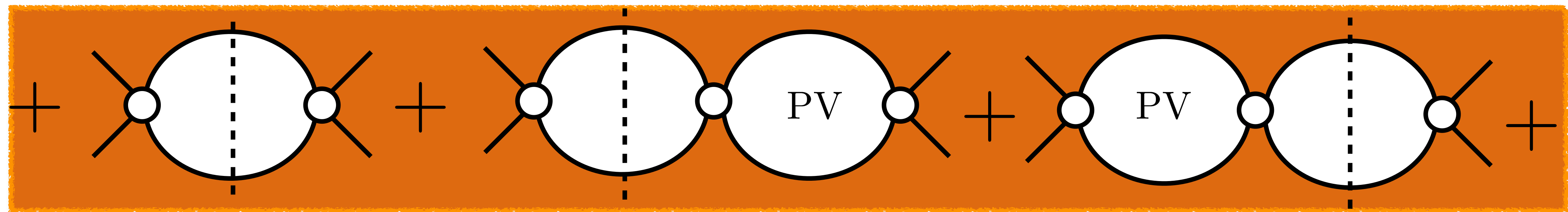
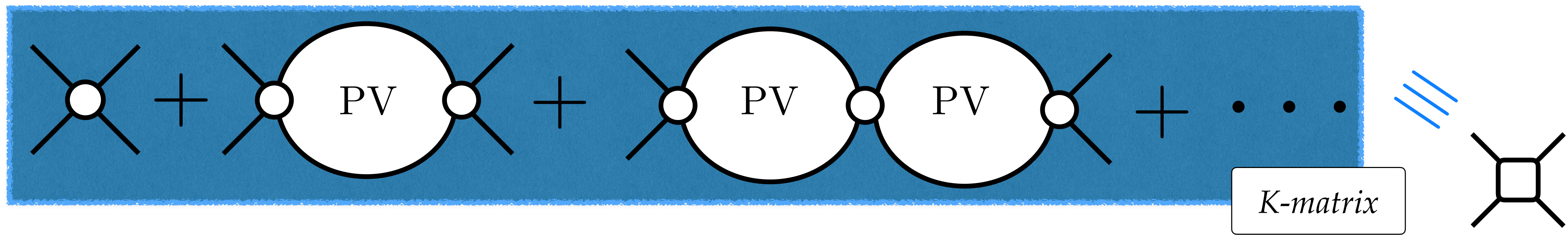
K-matrix



$$i\mathcal{M}_2 =$$



$$i\mathcal{M}_2 =$$



$$i\mathcal{M}_2 = \text{[Diagrammatic Series]} \equiv \text{[K-matrix Diagram]}$$

The first row shows a series of diagrams in a blue box. It starts with a four-point vertex, followed by a plus sign, a diagram with a circle labeled 'PV' between two vertices, another plus sign, a diagram with two circles labeled 'PV' between three vertices, and finally a plus sign followed by three dots. To the right of the blue box is a box labeled 'K-matrix' with three blue lines pointing to a square diagram with four external lines.

$$+ \text{[Diagrammatic Series]} +$$

The second row shows a series of diagrams in an orange box. It starts with a plus sign, followed by a diagram with a circle containing a vertical dashed line between two vertices, another plus sign, a diagram with a circle containing a vertical dashed line and a circle labeled 'PV' between three vertices, another plus sign, a diagram with a circle labeled 'PV' and a circle containing a vertical dashed line between three vertices, and finally a plus sign.

$$+ \text{[Diagrammatic Series]} + \dots$$

The third row shows a series of diagrams in an orange box. It starts with a plus sign, followed by a diagram with two circles containing vertical dashed lines between three vertices, another plus sign, and finally three dots.

$$\rho \equiv \frac{p}{8\pi E}$$

$$= \text{[Diagrammatic Series]}$$

The fourth row shows a series of diagrams in a red box. It starts with a plus sign, followed by a square diagram with four external lines, another plus sign, a diagram with a square and a circle containing a vertical dashed line between three vertices, another plus sign, a diagram with two squares and a circle containing a vertical dashed line between four vertices, another plus sign, a diagram with a square and a circle containing a vertical dashed line and a square between four vertices, another plus sign, and finally three dots.

$$= \frac{i}{\mathcal{K}^{-1} - i\rho}$$

A blue arrow points from the denominator of the fraction to the right.

Three-hadron systems

Three-hadron systems

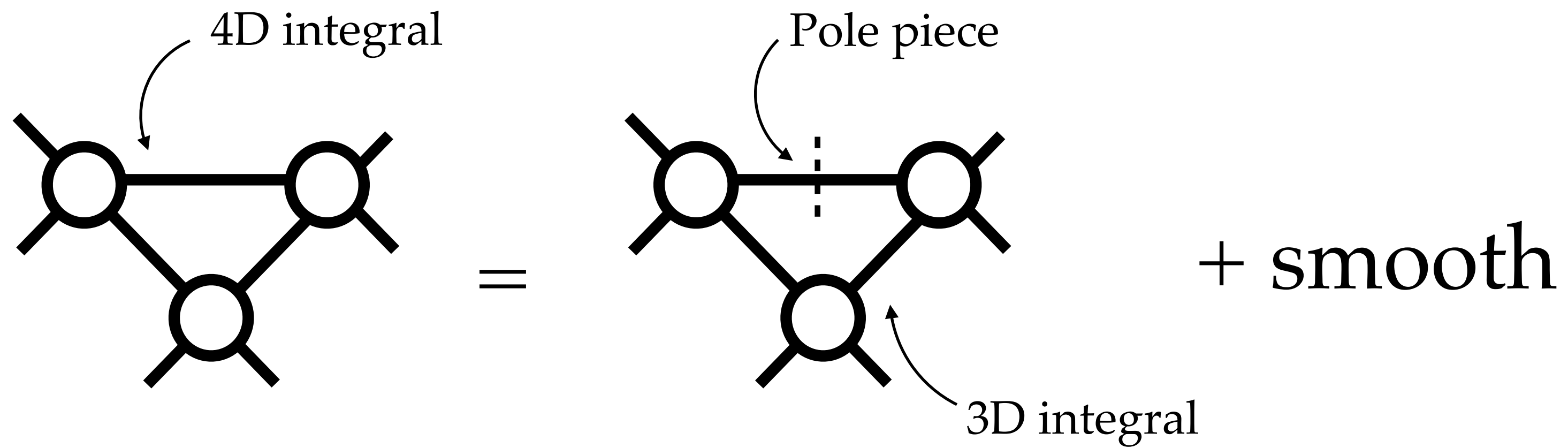
- Sum over all **3** \rightarrow **3** amputated diagrams

$$i\mathcal{M}_3 = \text{[Sun diagram]} = \text{[Tree diagram]} + \text{[Triangle diagram]} + \dots$$

The equation shows the decomposition of the three-hadron amplitude $i\mathcal{M}_3$. On the left is a sun diagram (a solid black circle with six external lines). This is equal to the sum of three diagrams: a tree diagram with two vertices, a triangle diagram with three vertices, and an ellipsis indicating further terms.

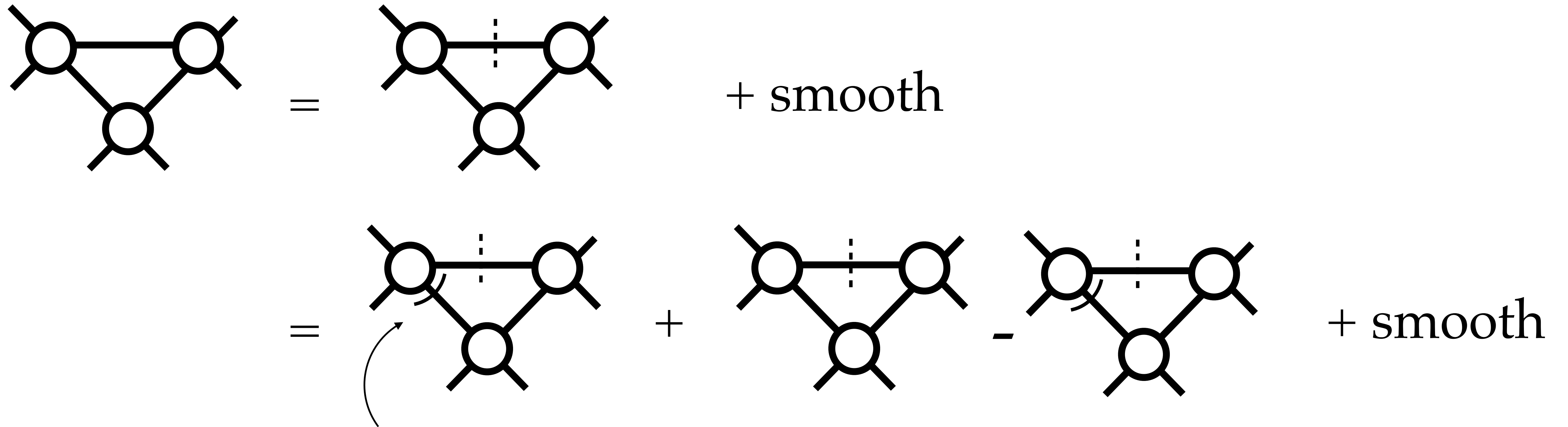
Reducing 4D to 3D

- ❑ Reducing from 4D to 3D while preserving singularities
- ❑ Remember, physical singularities are due to on-shell intermediate particles
- ❑ Let's consider a useful example:



Reducing 4D to 3D

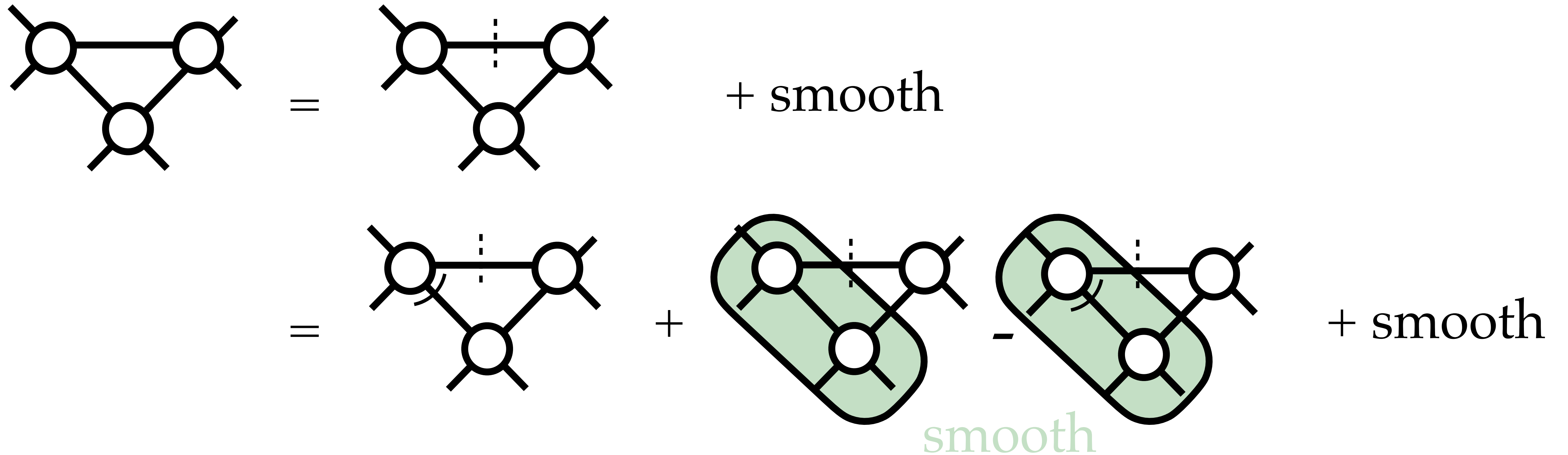
- ▣ Reducing from 4D to 3D while preserving singularities
- ▣ Remember, physical singularities are due to on-shell intermediate particles
- ▣ Let's consider a useful example:



place the vertex on shell
but not the pole...i.e. it's still a 3D integral.

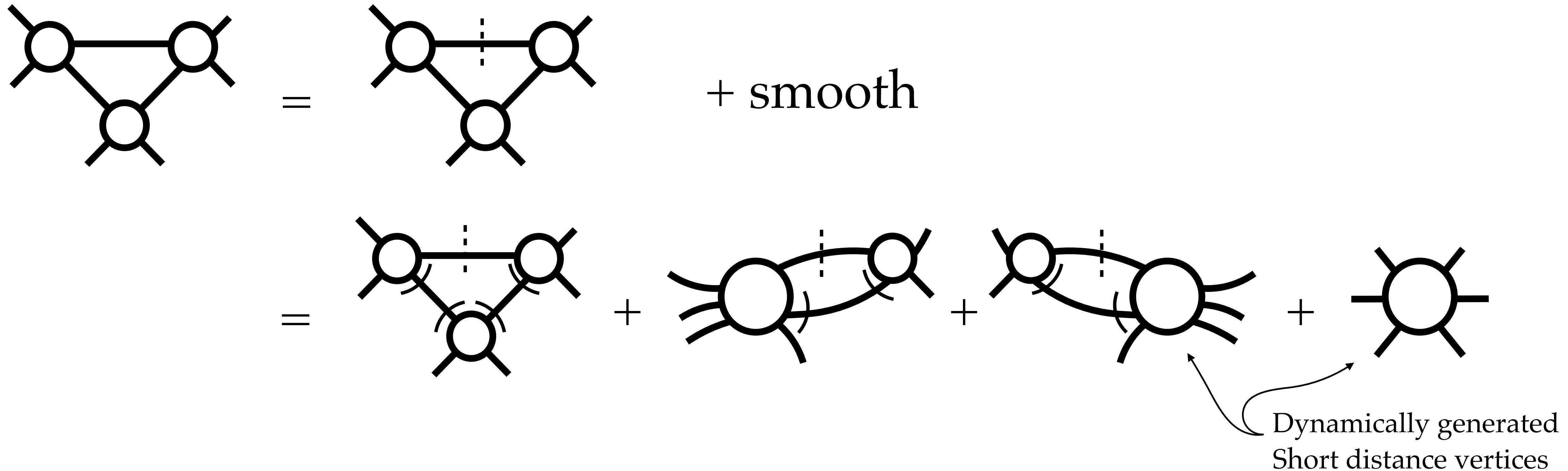
Reducing 4D to 3D

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Reducing 4D to 3D

- ▣ Reducing from 4D to 3D while preserving singularities
- ▣ Remember, physical singularities are due to on-shell intermediate particles
- ▣ Let's consider a useful example:



Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$i\mathcal{M}_3 = \text{[Diagram 1]} = \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} + \text{[Diagram 6]}$$

Scheme-dependent 3body K matrix

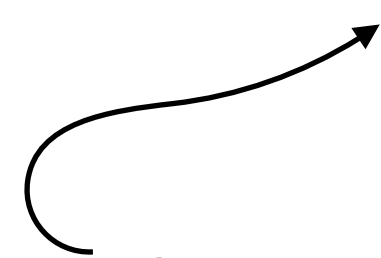
3D integrals with all vertices on-shell... will generally require a cutoff

Three-hadron systems

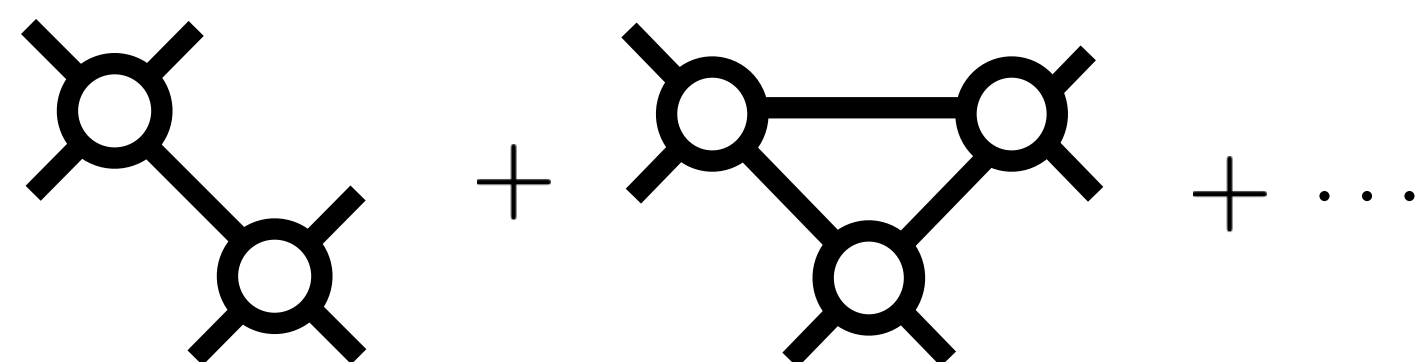
□ After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$i\mathcal{M}_3 = \text{[Diagram: 6 external lines meeting at a central black dot]} = \text{[Diagram: 6 external lines meeting at a central white circle]} + \text{[Diagram: Ladder diagram with 2 white circles and 1 black dot]} + \text{[Diagram: Ladder diagram with 1 white circle and 2 black dots]} + \text{[Diagram: Ladder diagram with 2 white circles and 1 black dot]} + \text{[Diagram: Ladder diagram with 1 white circle and 2 black dots]} + \text{[Diagram: Ladder diagram with 2 white circles and 1 black dot]}$$

$$= i\mathcal{D} + i\mathcal{M}_{3,\text{df}}$$



Sum over all ladder diagrams



Three-hadron systems

□ After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$i\mathcal{M}_3 = \text{[Diagram 1]} = \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} + \text{[Diagram 6]}$$

$$= i\mathcal{D} + i\mathcal{M}_{3,\text{df}}$$

↖
The rest [df = "divergence free"]

Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$i\mathcal{M}_3 = \text{[Diagram: 6 external lines meeting at a central black dot]} = \text{[Diagram: 6 external lines meeting at a central white circle]} + \text{[Diagram: 6 external lines, one white circle, one black dot]} + \text{[Diagram: 6 external lines, two white circles, one black dot]} + \text{[Diagram: 6 external lines, one white circle, one black dot, one loop]} + \text{[Diagram: 6 external lines, one white circle, one black dot, one loop]} + \text{[Diagram: 6 external lines, one white circle, one black dot, two loops]}$$
$$= i\mathcal{D} + i\mathcal{M}_{3,\text{df}}$$

- Satisfies integral equation

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$i\mathcal{M}_3 = \text{[Diagram: solid black circle with 6 external lines]} = \text{[Diagram: white circle with 6 external lines]} + \text{[Diagram: white circle with 3 external lines connected to a solid black circle with 3 external lines]} + \text{[Diagram: white circle with 3 external lines connected to a solid black circle with 3 external lines via a loop]} + \text{[Diagram: white circle with 3 external lines connected to a solid black circle with 3 external lines via a loop with a dot]} + \text{[Diagram: white circle with 3 external lines connected to a solid black circle with 3 external lines via a loop with a dot and a line]} + \text{[Diagram: white circle with 3 external lines connected to a solid black circle with 3 external lines via a loop with a dot and a line, and a bubble]}$$

$$= i\mathcal{D} + i\mathcal{M}_{3,\text{df}}$$

- Satisfies integral equation

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$\begin{aligned}
 i\mathcal{M}_3 &= \text{[Diagram: Full amplitude]} = \text{[Diagram: } i\mathcal{D}\text{]} + \text{[Diagram: } i\mathcal{M}_{3,\text{df}}\text{]} \\
 &= i\mathcal{D} + i\mathcal{M}_{3,\text{df}}
 \end{aligned}$$

- Satisfies integral equation

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

$$\mathcal{M}_{3,\text{df}}(\mathbf{p}, \mathbf{k}) = \int_{p'} \int_{k'} \mathcal{L}(\mathbf{p}, \mathbf{p}') \cdot \mathcal{T}(\mathbf{p}', \mathbf{k}') \cdot \mathcal{L}(\mathbf{k}', \mathbf{k}) \quad \Bigg| \quad \mathcal{L} = \frac{1}{3} + \mathcal{M}_2 \rho - \mathcal{D} \rho$$

Three-hadron systems

- Sum over all $3 \rightarrow 3$ amputated diagrams

$$i\mathcal{M}_3 = \text{[Diagram: 3 external lines meeting at a solid black vertex]} = \text{[Diagram: 3 external lines meeting at an open white vertex]} \\ = i\mathcal{D} + i\mathcal{M}_{3,\text{df}}$$

- Satisfies integral equation

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

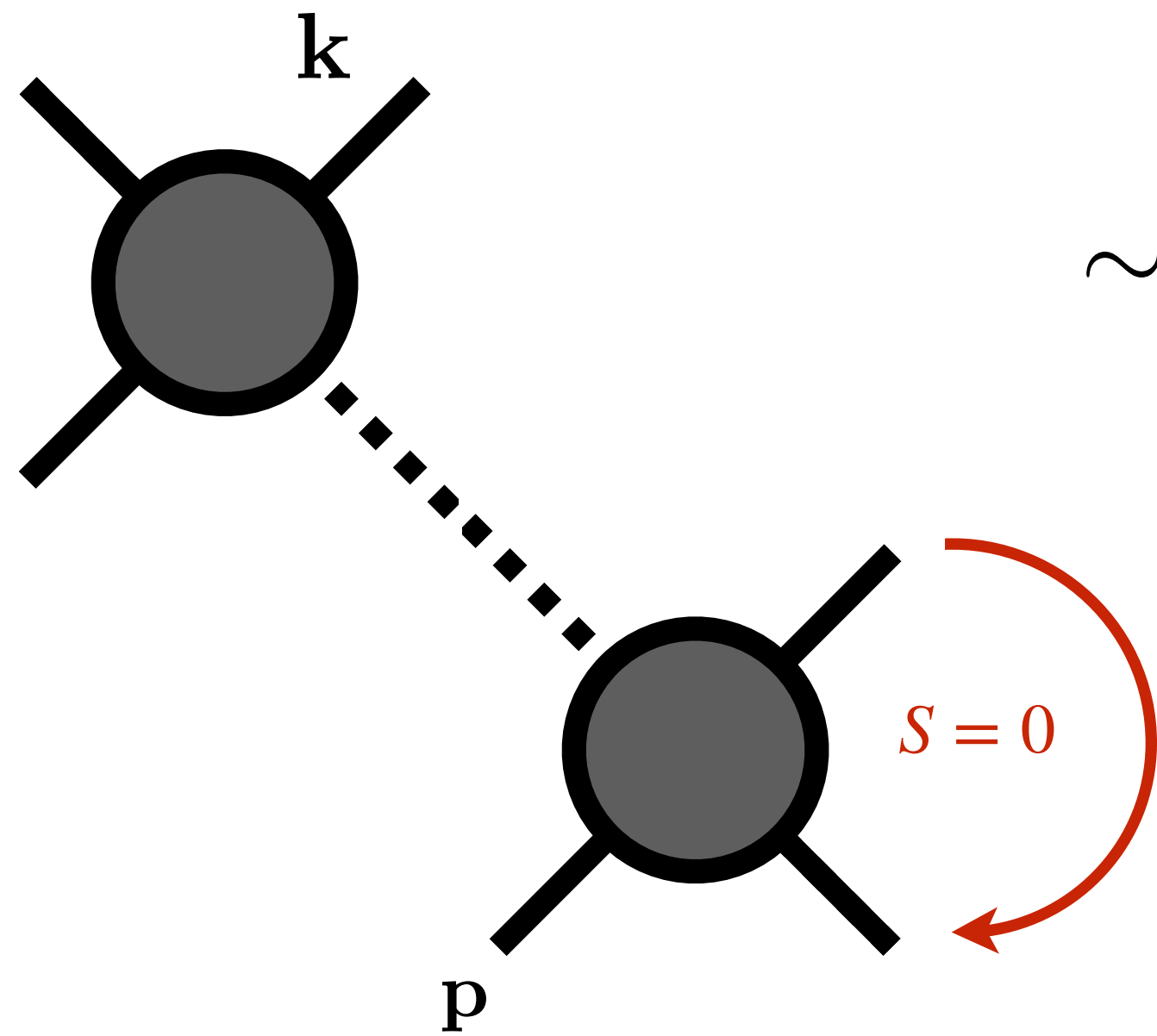
$$\mathcal{M}_{3,\text{df}}(\mathbf{p}, \mathbf{k}) = \int_{p'} \int_{k'} \mathcal{L}(\mathbf{p}, \mathbf{p}') \cdot \mathcal{T}(\mathbf{p}', \mathbf{k}') \cdot \mathcal{L}(\mathbf{k}', \mathbf{k}) \quad \Big| \quad \mathcal{L} = \frac{1}{3} + \mathcal{M}_2 \rho - \mathcal{D} \rho$$

$$\mathcal{T}(\mathbf{p}, \mathbf{k}) = \mathcal{K}_3(\mathbf{p}, \mathbf{k}) = \int_{p'} \int_{k'} \mathcal{K}_3(\mathbf{p}, \mathbf{p}') \cdot \frac{\rho(p')}{2\omega_{p'}} \mathcal{L}(\mathbf{p}', \mathbf{k}') \cdot \mathcal{T}(\mathbf{k}', \mathbf{k})$$

- Need to numerical solve these equations
 - Note: \mathcal{D} and \mathcal{T} are 3D integrals equations
 - Need to project to definite angular momentum and parity
 - Integration kernel generally singular
 - Singular behavior mainly driven by the OPE propagator, \mathcal{G}

OPE Partial wave projection

- Assuming s-wave two-body scattering



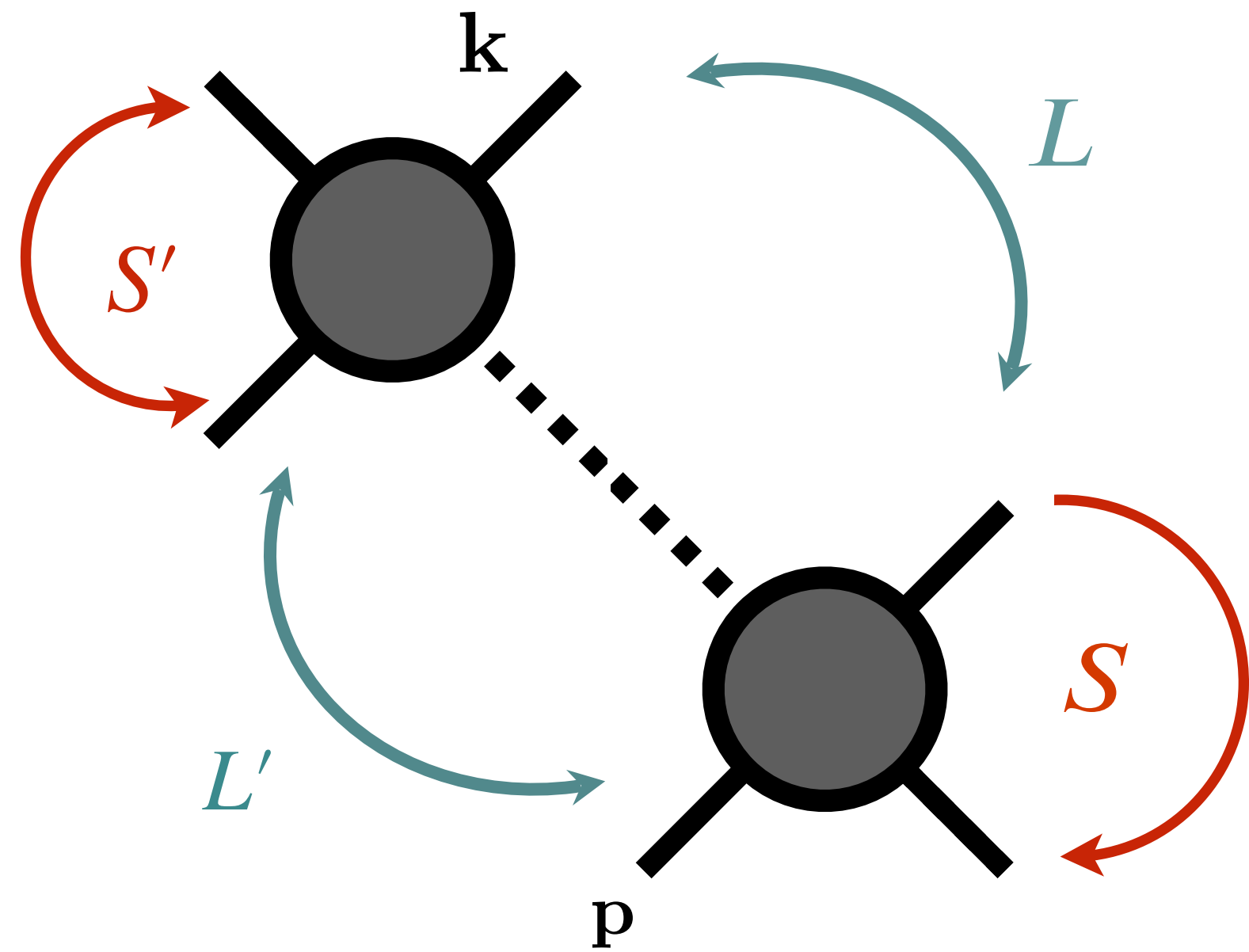
$$\begin{aligned} \sim G(\mathbf{p}, \mathbf{k}) &= \frac{1}{(E - \omega_k - \omega_p) - (\mathbf{p} + \mathbf{k})^2 - m^2 + i\epsilon} \\ &= \frac{1}{(E - \omega_k - \omega_p) - k^2 - p^2 - m^2 - 2pk \cos \theta + i\epsilon} \end{aligned}$$

- After partial wave projecting to total $J = 0$:

$$\sim G(p, k) = \frac{1}{2} \int_{-1}^1 d \cos \theta G(\mathbf{p}, \mathbf{k}) = -\frac{1}{4pk} \log \frac{z_{pk} - 1}{z_{pk} + 1}$$

Non-zero angular momentum

- In general...assuming spinless particles

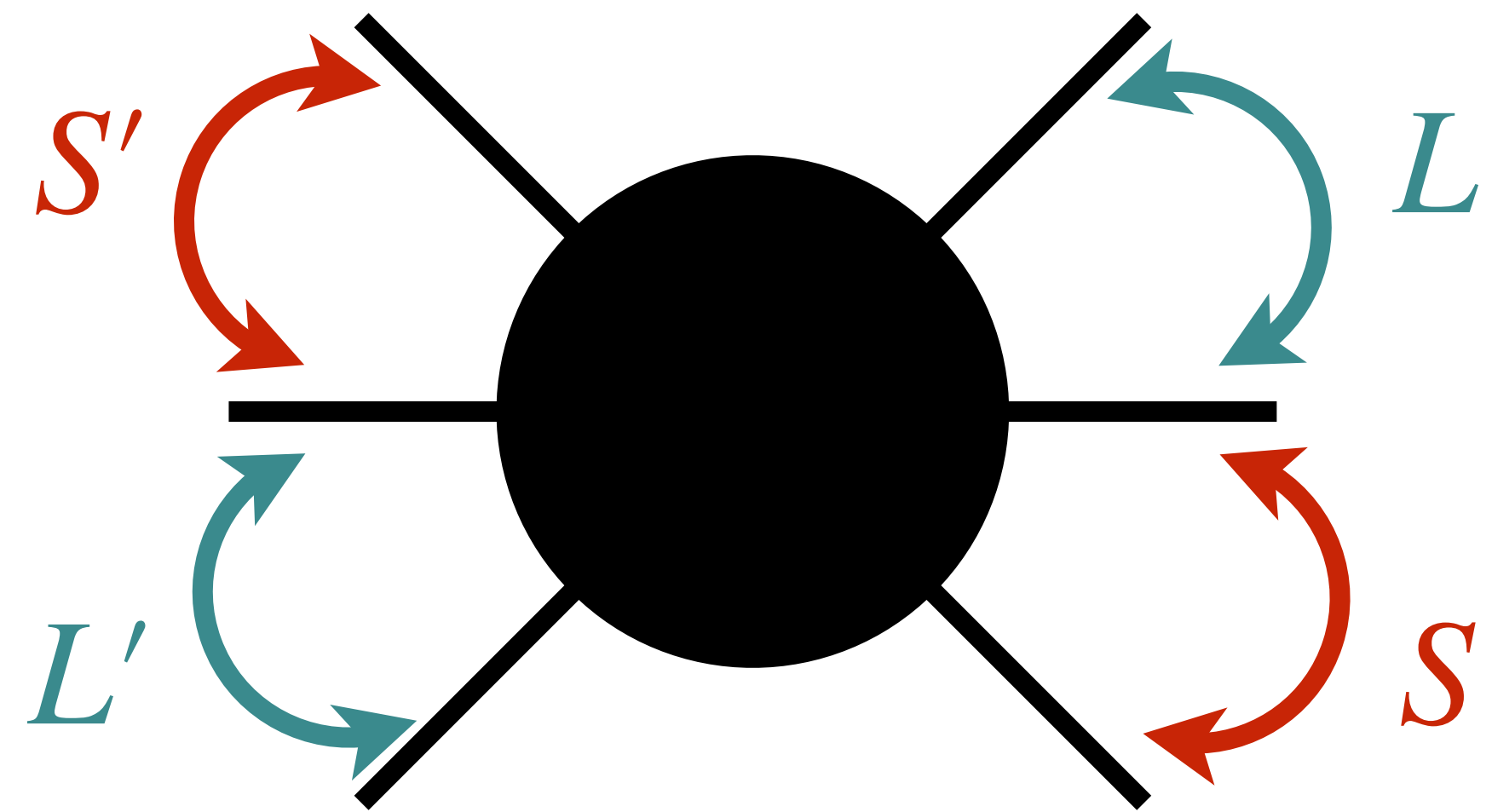


$$\mathcal{G}_{L'S',LS}^{JP}(p, k) = \underbrace{\mathcal{K}_{\mathcal{G};L'S',LS}^{JP}(p, k)}_{\text{known kinematic functions}} + \underbrace{\mathcal{C}_{L'S',LS}^{JP}(p, k)}_{\text{non-analyticity entirely encoded in}} \underbrace{Q_0(\zeta_{pk})}_{\text{non-analyticity entirely encoded in}}$$

- $\mathcal{K}_{\mathcal{G};L'S',LS}^{JP}$ and $\mathcal{C}_{L'S',LS}^{JP}(p, k)$ are known kinematic functions that need to be generated for each channel

Partial wave projections

□ In general:



The diagram shows a central black circle representing a scattering potential. Four black lines represent incoming and outgoing waves. On the left, two lines are labeled L' (teal) and S' (red). On the right, two lines are labeled L (teal) and S (red). Curved arrows indicate the angular momentum of each wave: L' and L are teal and point counter-clockwise, while S' and S are red and point clockwise.

$$= i \left[\mathcal{M}_3^{J^P} \right]_{L' S', L S}$$

Partial wave projections

- Partial wave projected amplitude

$$\mathcal{M}_3^{JP}(p, k) = \mathcal{D}^{JP}(p, k) + \mathcal{M}_{3,\text{df}}^{JP}(p, k)$$

$$\mathcal{D}^{JP}(p, k) = \mathcal{D}_0^{JP}(p, k) - \mathcal{M}_2(\sigma_p) \cdot \int_{k'} \mathcal{G}^{JP}(p, k') \cdot \mathcal{D}^{JP}(k', k)$$

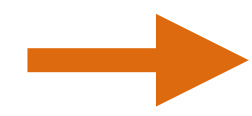
- Two classes of K matrix are possible

1. Symmetric K matrix \rightarrow $\mathcal{M}_{3,\text{df}}^{JP}(p, k) = \int_{p'} \int_{k'} \mathcal{L}^{JP}(p, p') \cdot \mathcal{T}^{JP}(p', k') \cdot \mathcal{R}^{JP}(k', k)$
2. Asymmetric K matrix \rightarrow $\widehat{\mathcal{M}}_{3,\text{df}}^{JP}(p, k) = \int_{p'} \int_{k'} \widehat{\mathcal{L}}^{JP}(p, p') \cdot \widehat{\mathcal{T}}^{JP}(p', k') \cdot \widehat{\mathcal{R}}^{JP}(k', k)$

Partial wave projections

□ Two classes of K matrix are possible

2. Asymmetric K matrix



$$\widehat{\mathcal{M}}_{3,\text{df}}^{JP}(p, k) = \int_{p'} \int_{k'} \widehat{\mathcal{L}}^{JP}(p, p') \cdot \widehat{\mathcal{T}}^{JP}(p', k') \cdot \widehat{\mathcal{R}}^{JP}(k', k)$$

$$\left[\widehat{\mathcal{L}}^{JP}(p, k) \right]_{L'S', LS} = [1 - \mathcal{M}_{2,S'}(\sigma_p) \tilde{\rho}(\sigma_p)] \delta_{L'L} \delta_{S'S} \frac{(2\pi)^2 \omega_k}{k^2} \delta(p - k) - \mathcal{M}_{2,S'}(\sigma_p) \mathcal{G}_{L'S', LS}^{JP}(p, k)$$

$$- \mathcal{D}_{L'S', LS}^{JP}(p, k) \tilde{\rho}(\sigma_k) - \sum_{L'', S''} \int_{k'} \mathcal{D}_{L'S', L''S''}^{JP}(p, k') \mathcal{G}_{L''S'', LS}^{JP}(k', k)$$

$$\left[\widehat{\mathcal{R}}^{JP}(p, k) \right]_{L'S', LS} = [1 - \tilde{\rho}(\sigma_k) \mathcal{M}_{2,S'}(\sigma_k)] \delta_{L'L} \delta_{S'S} \frac{(2\pi)^2 \omega_p}{p^2} \delta(p - k) - \mathcal{G}_{L'S', LS}^{JP}(p, k) \mathcal{M}_{2,S}(\sigma_k)$$

$$- \tilde{\rho}(\sigma_p) \mathcal{D}_{L'S', LS}^{JP}(p, k) - \sum_{L'', S''} \int_{p'} \mathcal{G}_{L'S', L''S''}^{JP}(p, p') \mathcal{D}_{L''S'', LS}^{JP}(p', k)$$

$$\widehat{\mathcal{T}}^{JP}(p, k) = \widehat{\mathcal{K}}_3^{JP}(p, k) - \int_{p'} \int_{k'} \widehat{\mathcal{K}}_3^{JP}(p, p') \cdot \widehat{\mathcal{F}}^{JP}(p', k') \cdot \widehat{\mathcal{T}}^{JP}(k', k) \quad \widehat{\mathcal{F}}^{JP}(p, k) \equiv \tilde{\rho}(\sigma_p) \widehat{\mathcal{L}}^{JP}(p, k) + \int_{k'} \mathcal{G}(p, k') \cdot \widehat{\mathcal{L}}^{JP}(k', k)$$

Separable K matrix

- We consider a parametrization for which the kinematic dependence factorizes:

$$\left[\widehat{\mathcal{K}}_3^{JP}(p, k) \right]_{L'S', LS} = [h(p)]_{L'S'} \left[\widetilde{\mathcal{K}}_3^{JP}(s) \right]_{L'S', LS} [h(k)]_{LS}$$

Separable K matrix

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leads to an also factorizable $\widehat{\mathcal{T}}^{JP}$

$$\widetilde{\mathcal{T}}^{JP}(s) = \frac{1}{1 + \widetilde{\mathcal{K}}_3^{JP}(s) \cdot \widetilde{\mathcal{F}}^{JP}(s)} \cdot \widetilde{\mathcal{K}}_3^{JP}(s)$$

$$\widetilde{\mathcal{F}}^{JP}(s) = \int_p \int_k \widetilde{\mathcal{R}}^{JP}(s, p) \cdot \Gamma^{JP}(p, k) \cdot h(k)$$

$$\Gamma^{JP}(p, k) = \frac{(2\pi)^2 \omega_k}{k^2} \delta(p - k) \widetilde{\rho}(\sigma_p) + \mathcal{G}^{JP}(p, k)$$

From 3body \rightarrow 2body

□ \mathcal{M}_3^{JP} is effectively described as a 2-body system (pair+spectator)

□ pair can become bound: $b_k + \varphi_k \rightarrow b_p + \varphi_p$

$$\mathcal{M}_2(\sigma_k) = \text{[Diagram: Black circle with four external lines]} \sim \text{[Diagram: Black rectangle with four external lines]} \sim -\frac{g_{k,b}^2}{\sigma_k - \sigma_{k,b}}$$

□ Bound-state spectator amplitude:

$$\mathcal{M}_3^{JP} \sim \left(-\frac{g_{p,b}}{\sigma_p - \sigma_{p,b}} \right) \text{[Diagram: Blue circle with four external lines, two blue and two black]} \left(-\frac{g_{k,b}}{\sigma_k - \sigma_{k,b}} \right)$$

$$\mathcal{M}_{\varphi b}^{JP} = \lim_{\substack{\sigma_p \rightarrow \sigma_{p,b} \\ \sigma_k \rightarrow \sigma_{k,b}}} \frac{(\sigma_p - \sigma_{p,b})(\sigma_k - \sigma_{k,b})}{g_{p,b}g_{k,b}} \mathcal{M}_3^{JP}$$

Toy models for 3π systems

$S \leq 1$ and $L \leq 2$

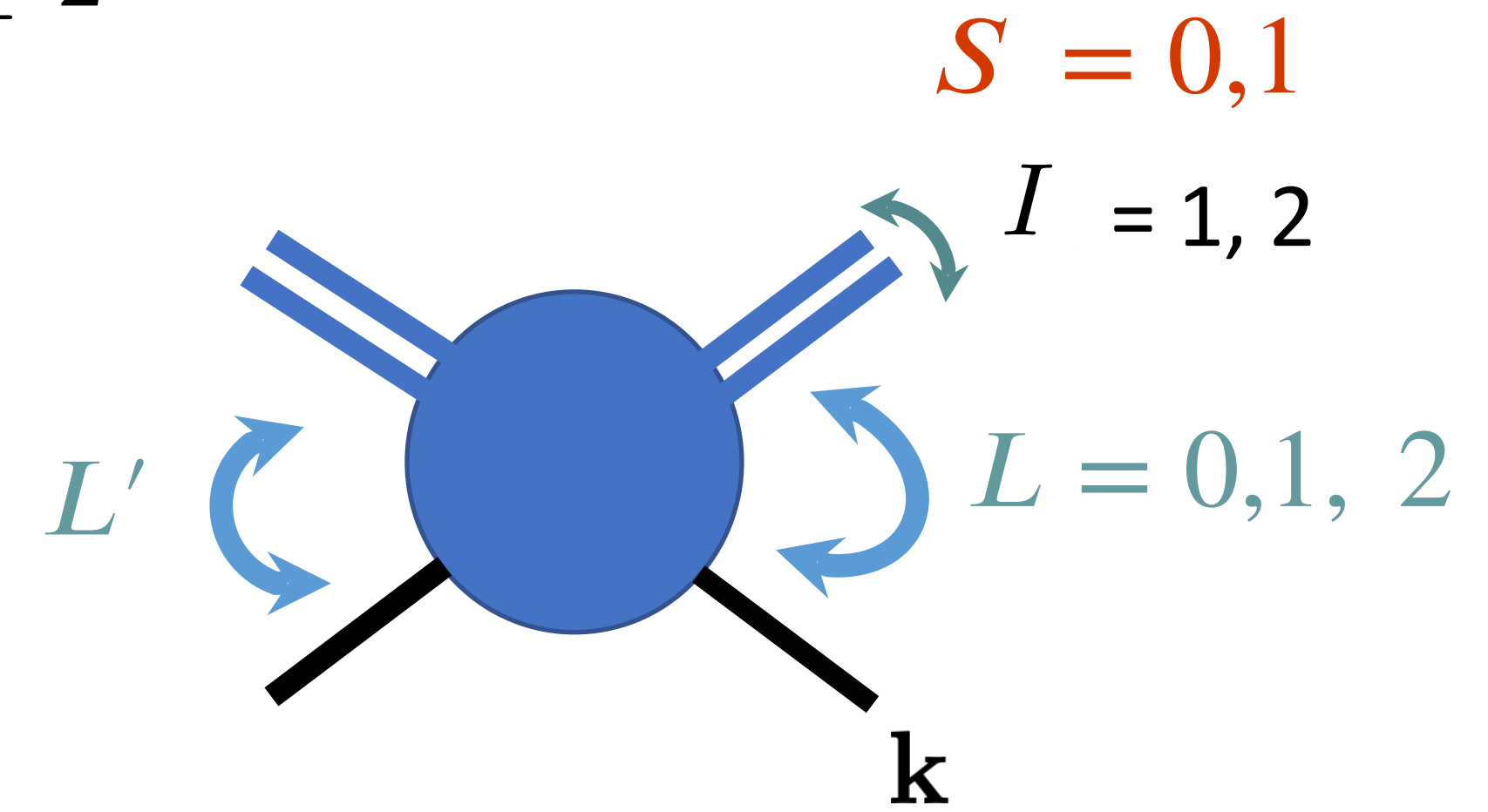
$$[\pi\pi]_{\ell}^I + \pi \rightarrow [\pi\pi]_{\ell'}^{I'} + \pi$$

Toy models for 3π systems

$$S \leq 1 \text{ and } L \leq 2$$

$$[\pi\pi]_{\ell}^I + \pi \rightarrow [\pi\pi]_{\ell'}^{I'} + \pi$$

$T(J^P) = 2(1^+)$ channel with stable ρ



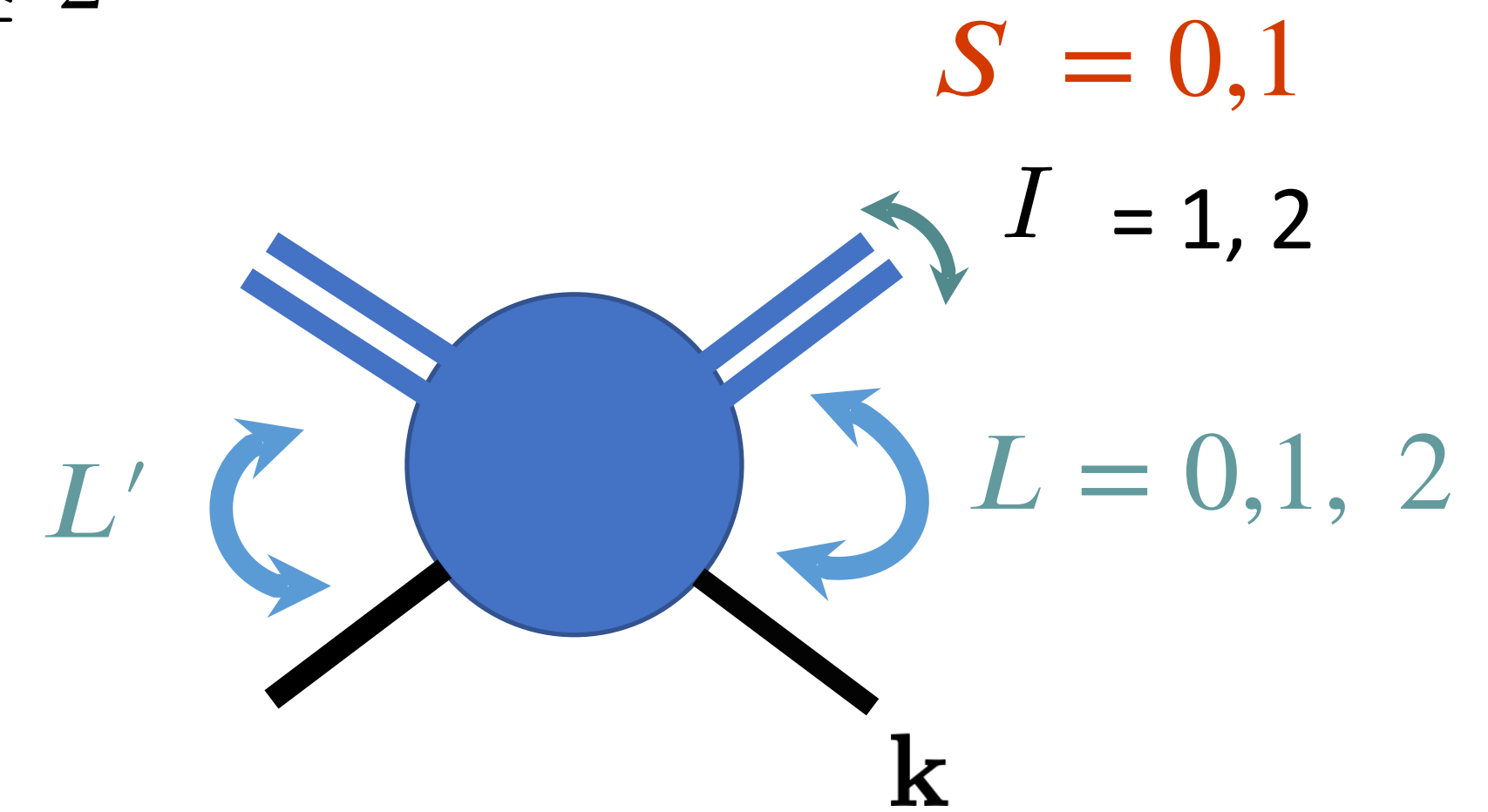
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$$([\pi\pi]_S^2\pi)_P, ([\pi\pi]_P^1\pi)_S, ([\pi\pi]_P^1\pi)_D$$

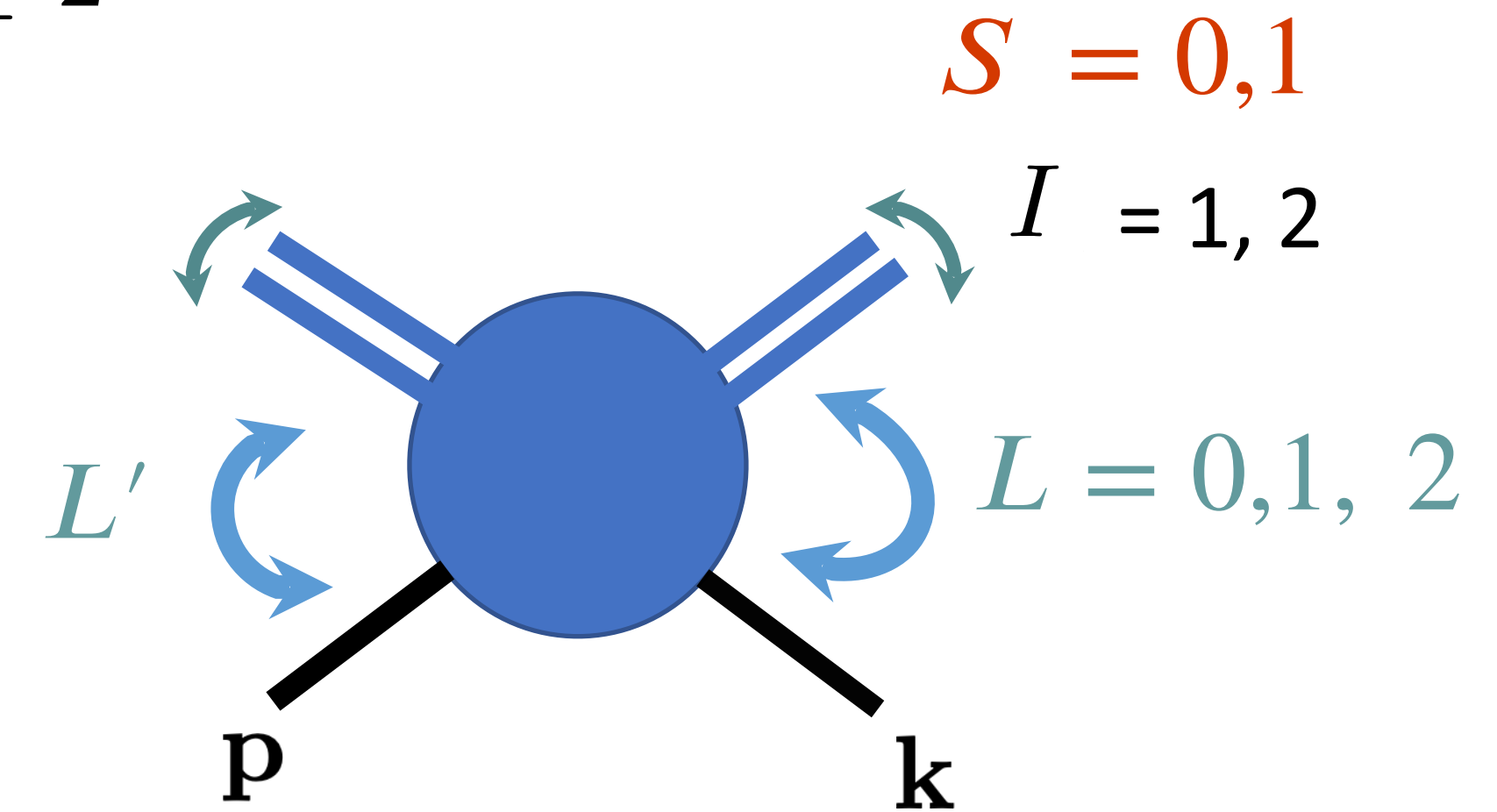


Toy models for 3π systems

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$$([\pi\pi]_S^2\pi)_P, ([\pi\pi]_P^1\pi)_S, ([\pi\pi]_P^1\pi)_D$$

Can couple to the rho

Bound state model

- Can get a two-body bound state for S- and P-waves by parametrizing \mathcal{M}_2 via the phase shift:

$$\mathcal{M}_{2,IS}(\sigma_k) = \frac{16\pi\sqrt{\sigma_k}}{q_k^* \cot \delta_{S,I} - iq_k^*} \quad \left(q_k^* = \sqrt{\sigma_k/4 - m^2} \right)$$

- For a S-wave 2 body amplitude, we can use a LO ERE:

$$q_k^* \cot \delta_{0,I} = -\frac{1}{a_{0,I}}$$

- For a P-wave bound state, LO ERE leads to unphysical poles

- Instead, for P-waves, we use

$$q_k^* \cot \delta_{1,1} = \frac{(m_{\text{BW}}^2 - \sigma_k)}{\sqrt{\sigma_k} \Gamma_1^{\text{BW}}(\sigma_k)} \quad \Gamma_1^{\text{BW}}(\sigma_k) = \frac{g_{\text{BW}}^2}{6\pi\sigma_k} q_k^{*2}$$

Solving integral equations

- Deform contour to miss singularities and discretize momenta
 - sometimes useful // sometimes critical

- Discretize momenta:
$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s) d(q, s, p)$$
$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\max}} K(p', s, q) d(q, s, p)$$

[contains pole, logarithmic and square root cuts]



Solving integral equations

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$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\max}} K(p', s, q) d(q, s, p)$$

- Use linear algebra:

$$[1 + \mathbf{K}] \cdot \vec{d}_{\text{sol}}(s, p) = -\vec{G}(s, p)$$

Solving integral equations

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$$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s) d(q, s, p)$$
$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\max}} K(p', s, q) d(q, s, p)$$

□ Use linear algebra:

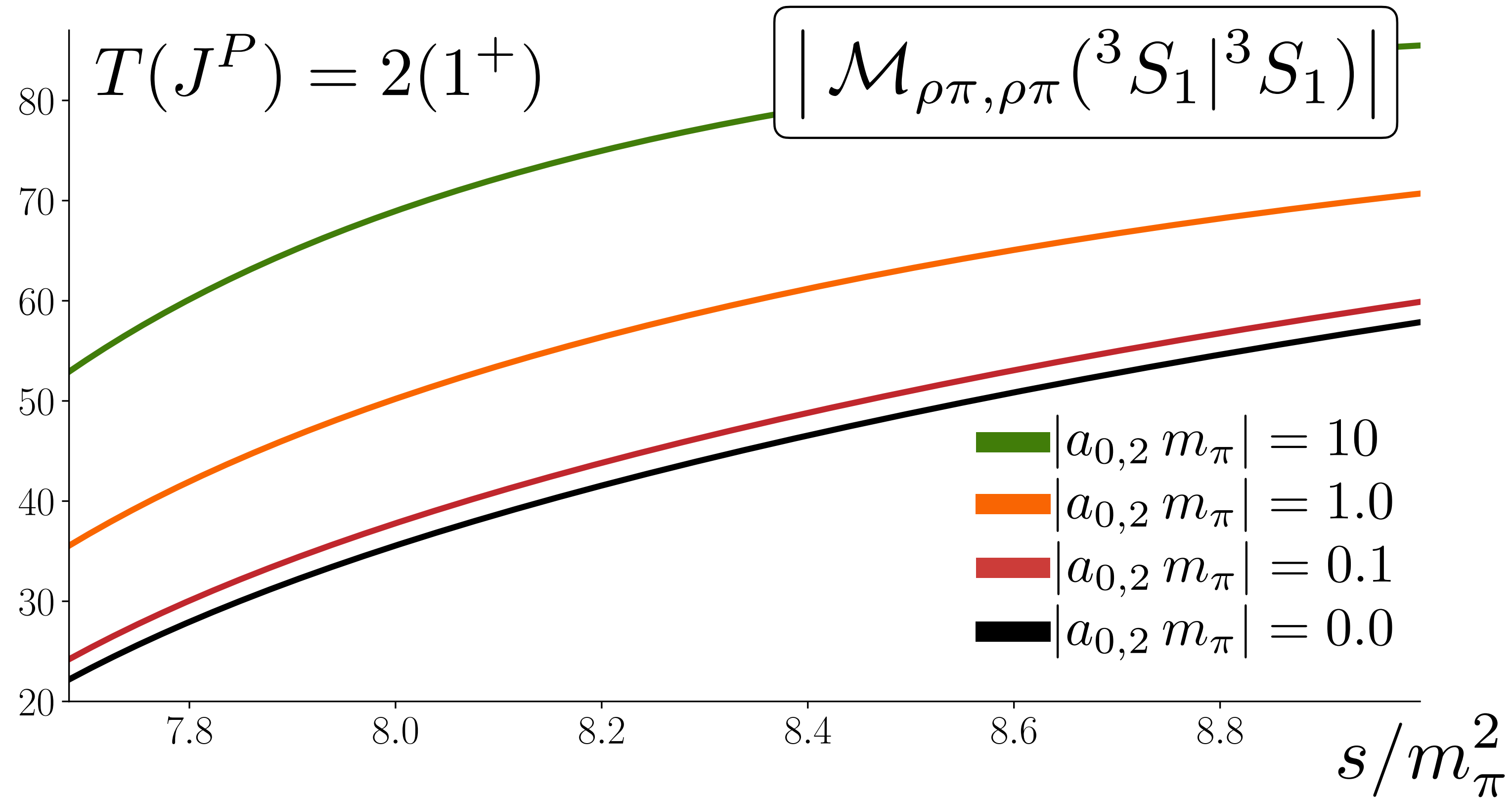
$$[1 + \mathbf{K}] \cdot \vec{d}_{\text{sol}}(s, p) = -\vec{G}(s, p)$$

□ Use integral equation to interpolate or extrapolate:

$$d(p', s, p) \approx -G(p', s, p) - \vec{K}(p', s) \cdot \vec{d}_{\text{sol}}(s, p)$$

$T(J^P) = 2(1^+)$ channel with stable ρ

$$d^{2(1^+)} = \begin{pmatrix} d_{\rho\pi,\rho\pi}({}^3S_1|{}^3S_1) & d_{\rho\pi,\rho\pi}({}^3S_1|{}^3D_1) & d_{\rho\pi,t\pi}({}^3S_1|{}^1P_1) \\ & d_{\rho\pi,\rho\pi}({}^3D_1|{}^3D_1) & d_{\rho\pi,t\pi}({}^3D_1|{}^1P_1) \\ & & d_{t\pi,t\pi}({}^1P_1|{}^1P_1) \end{pmatrix}$$



$T(J^P) = 2(1^+)$ channel with stable ρ

□ Setting $a_{0,2} = 0$

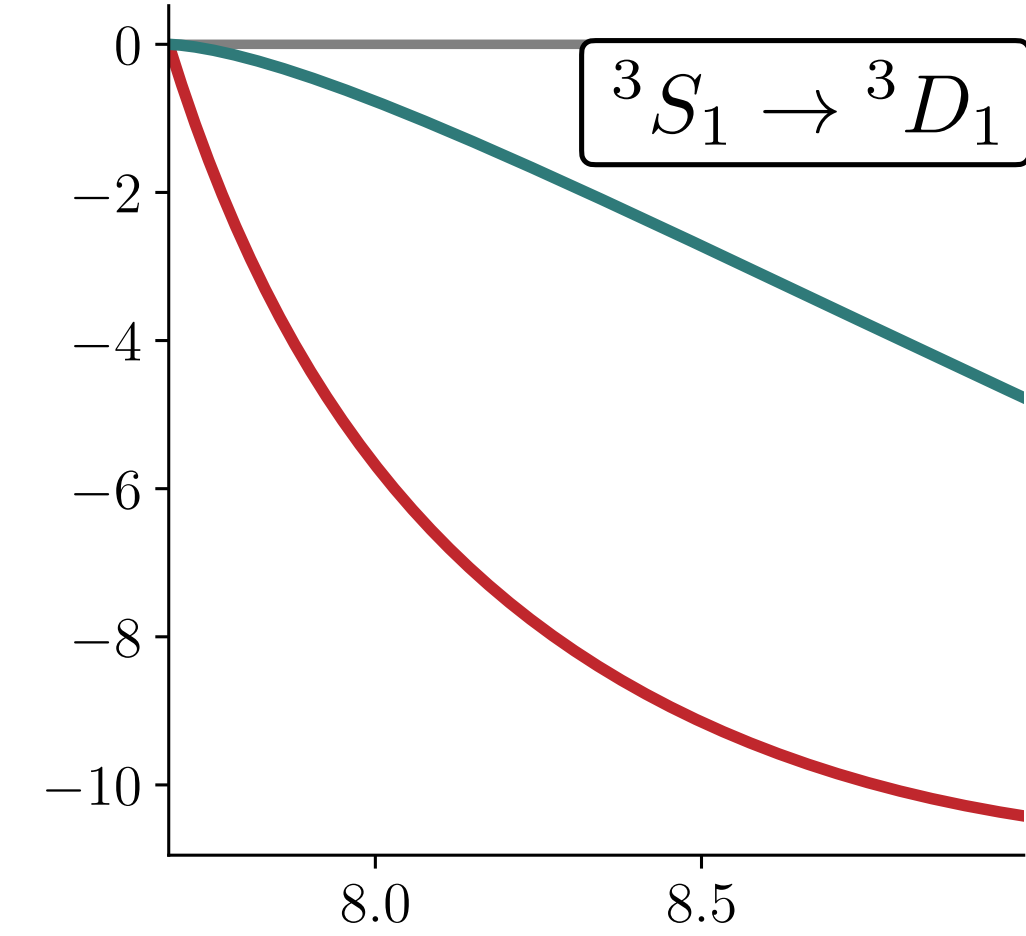
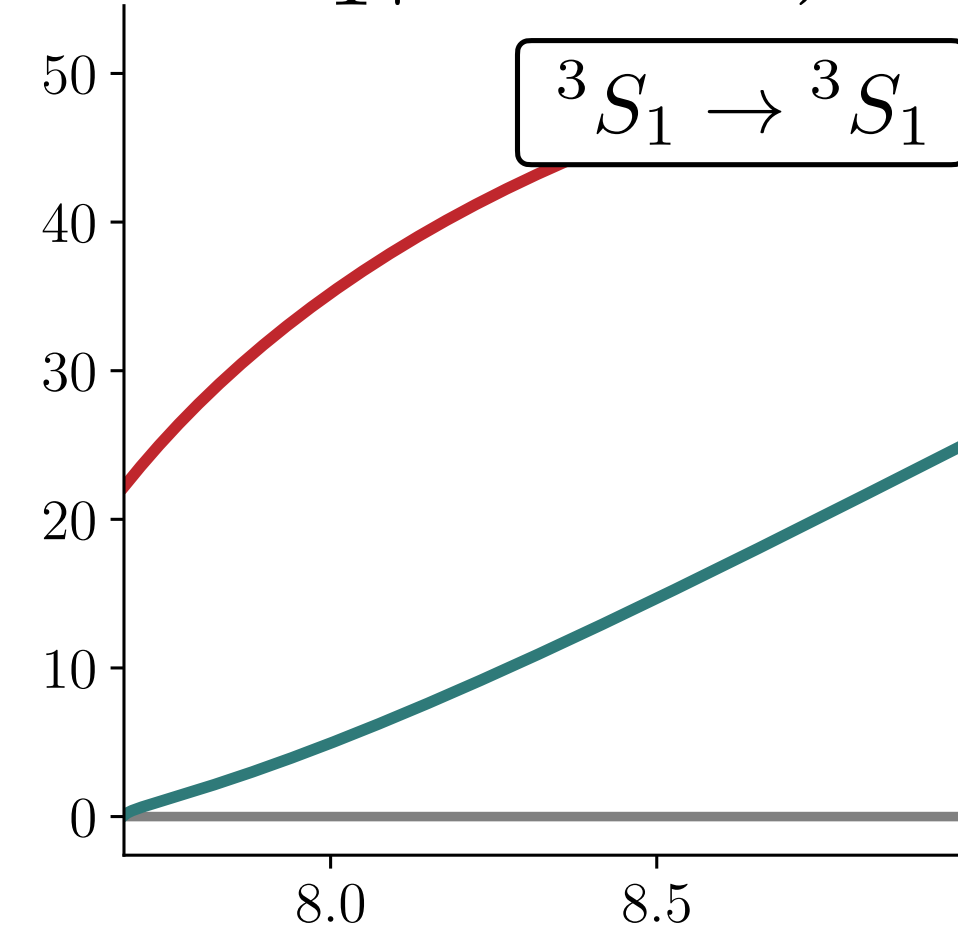
$$\mathcal{M}_{\varphi b}^{2(1^+)} = \begin{pmatrix} \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3S_1) & \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3D_1) \\ \mathcal{M}_{\rho\pi,\rho\pi}({}^3D_1|{}^3D_1) \end{pmatrix}$$

□ Satisfy 2-body unitarity condition

□ Satisfy threshold behavior:

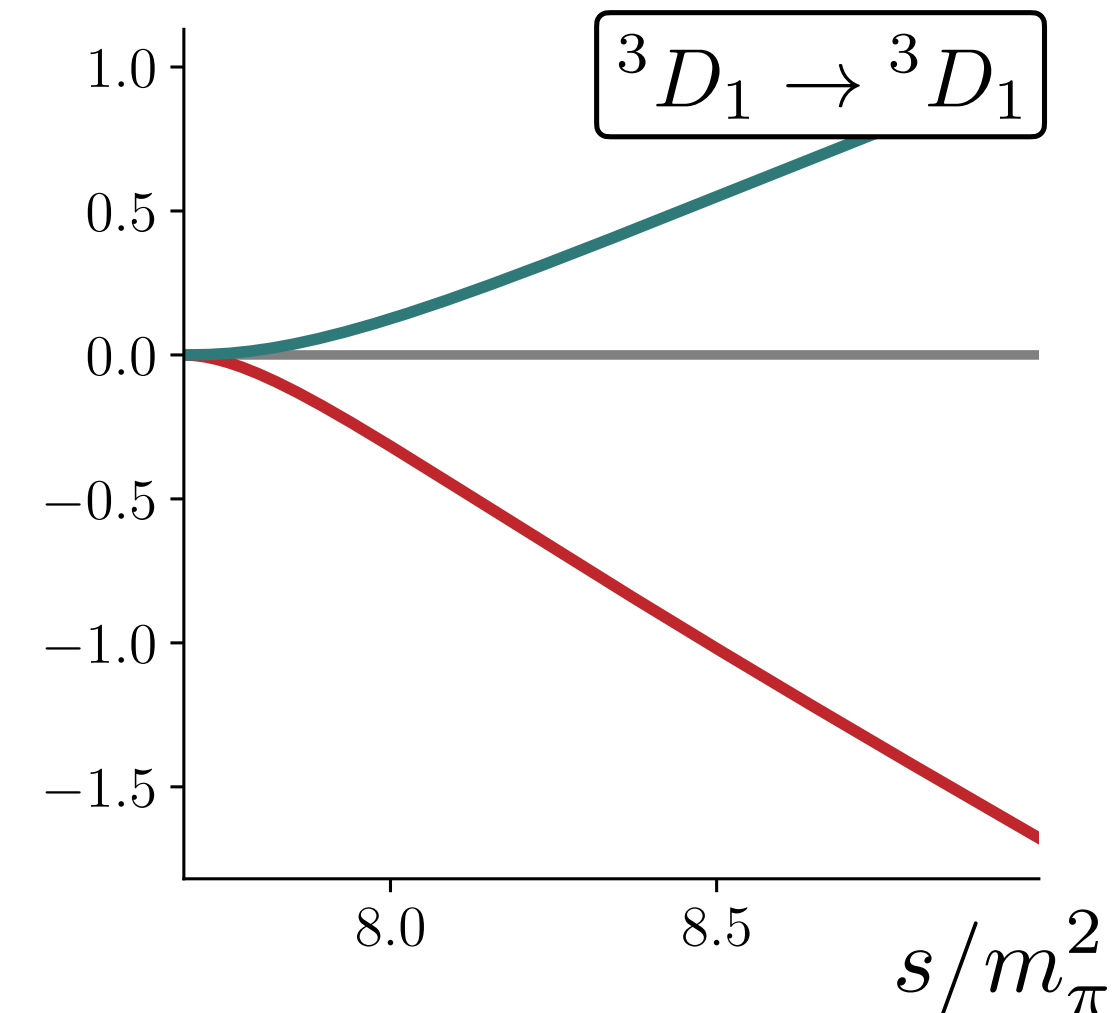
$$\mathcal{M}_{\rho\pi,\rho\pi}({}^3L'_1|{}^3L_1) \sim q_{\rho\pi}^{L'+L}$$

$$\mathcal{M}({}^{2S'+1}L'_1|{}^{2S+1}L_1)$$



$$T(J^P) = 2(1^+)$$

— real
— imag



$T(J^P) = 1(1^+)$ *channel with stable σ and ρ*

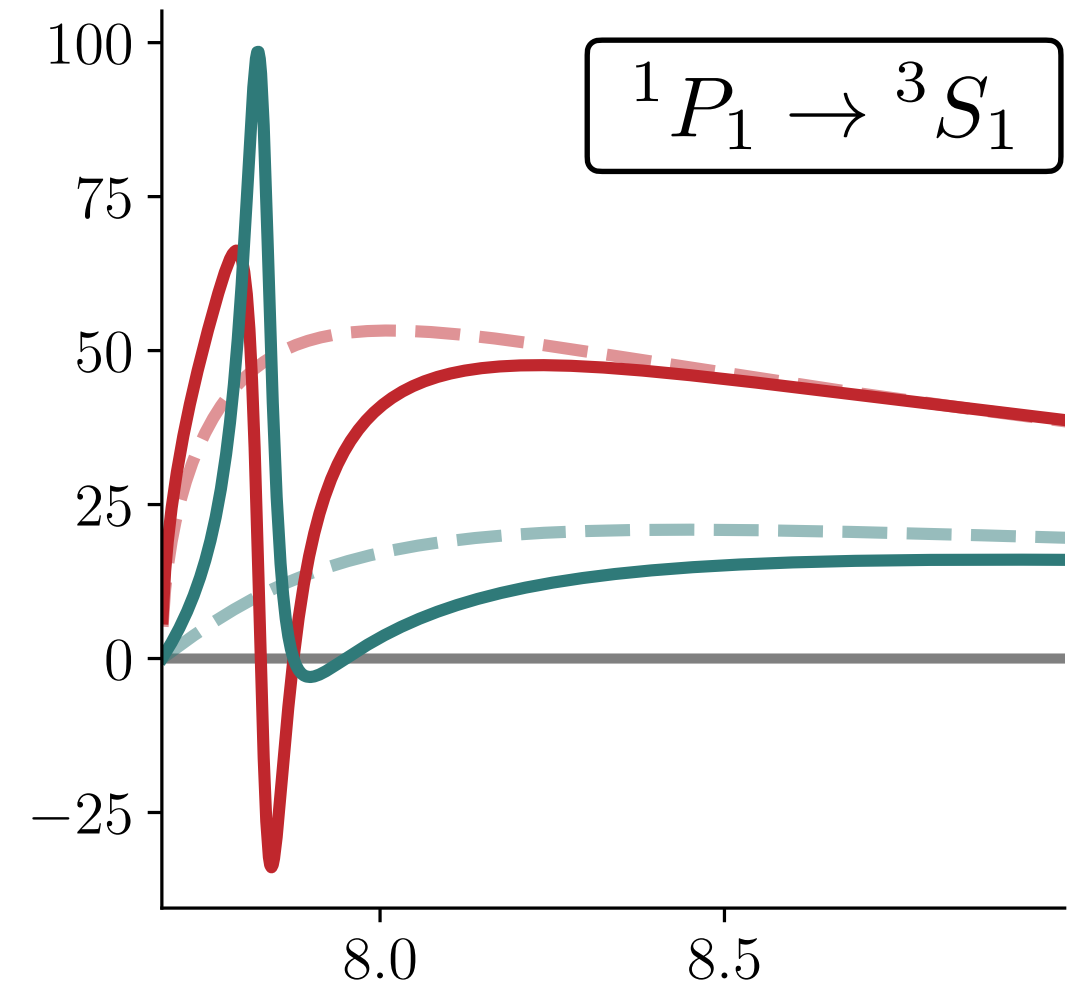
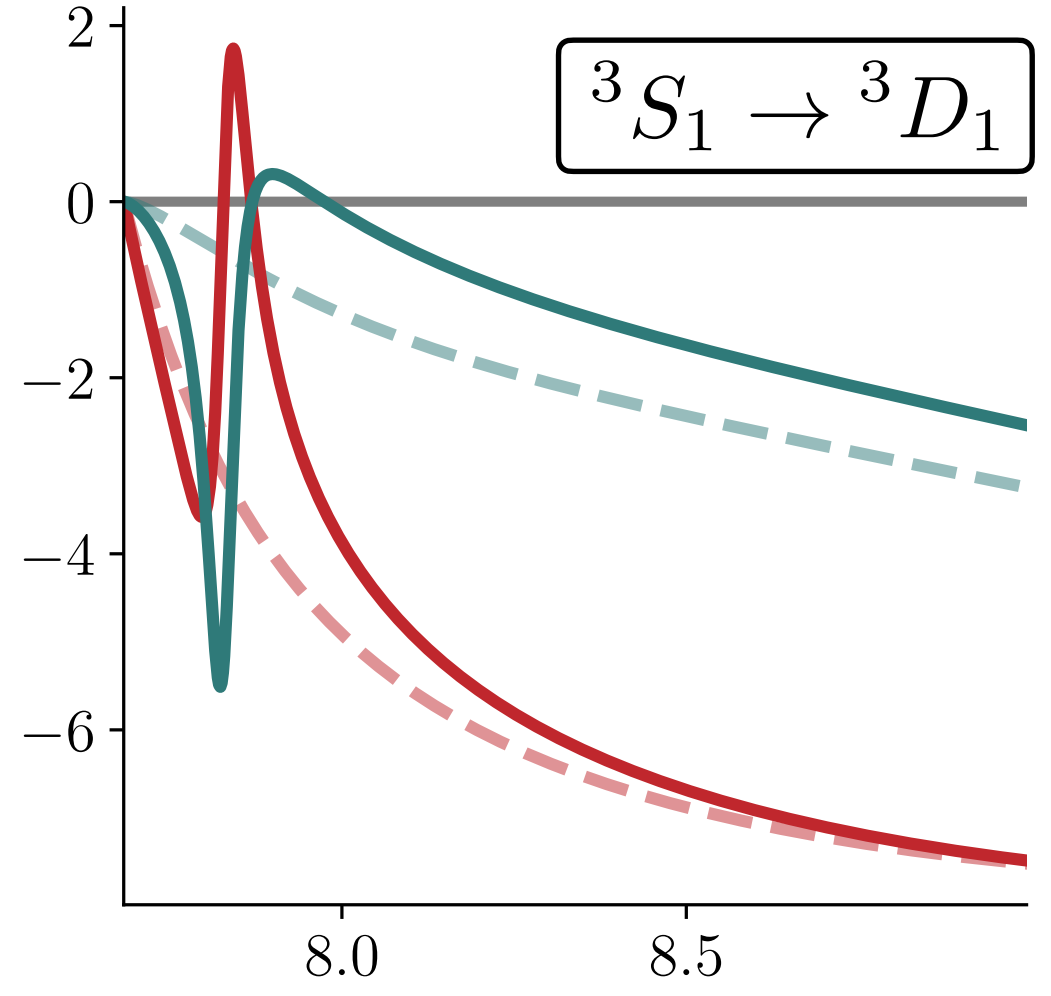
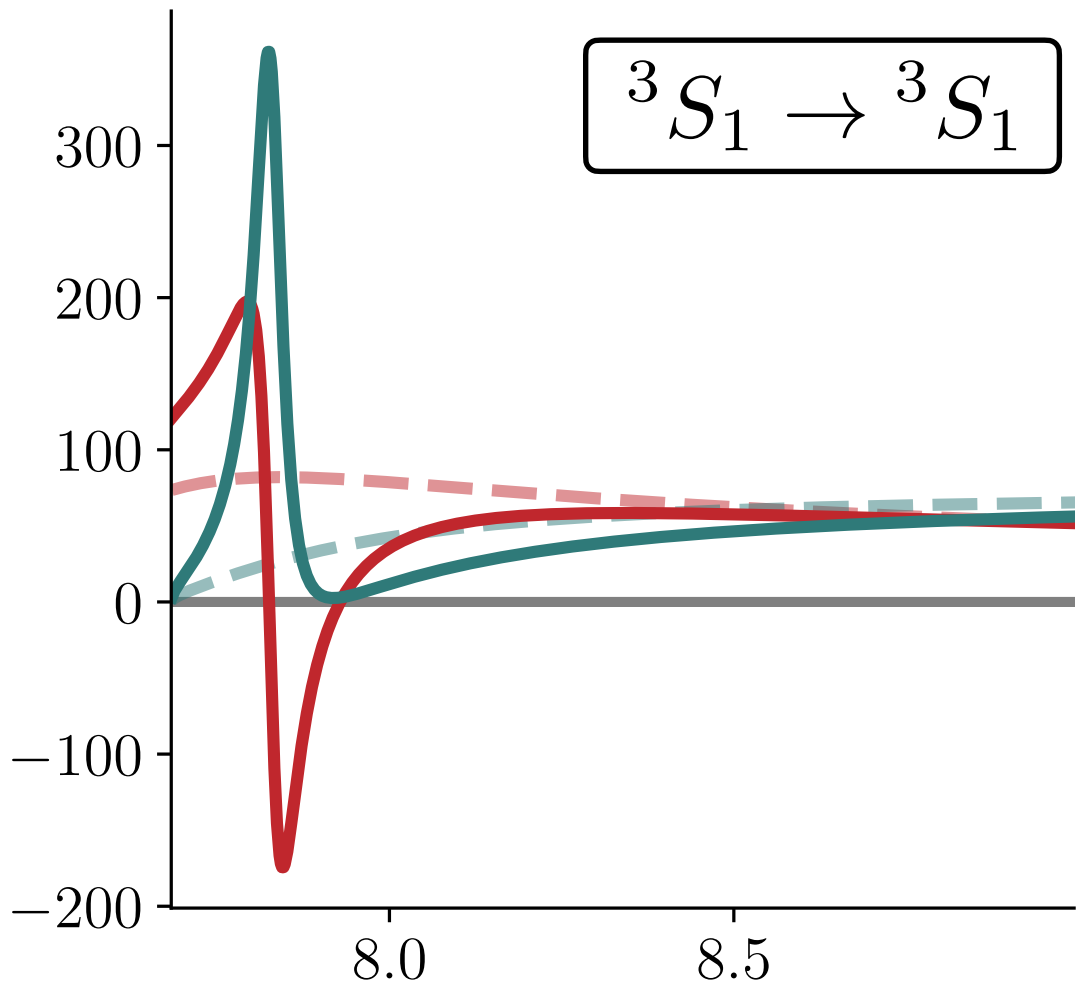
$([\pi\pi]_S^{0,2}\pi)_P, ([\pi\pi]_P^1\pi)_S, ([\pi\pi]_P^1\pi)_D$

$T(J^P) = 1(1^+)$ *channel with stable σ and ρ*

$([\pi\pi]_S^{0,2}\pi)_P, ([\pi\pi]_P^1\pi)_S, ([\pi\pi]_P^1\pi)_D$

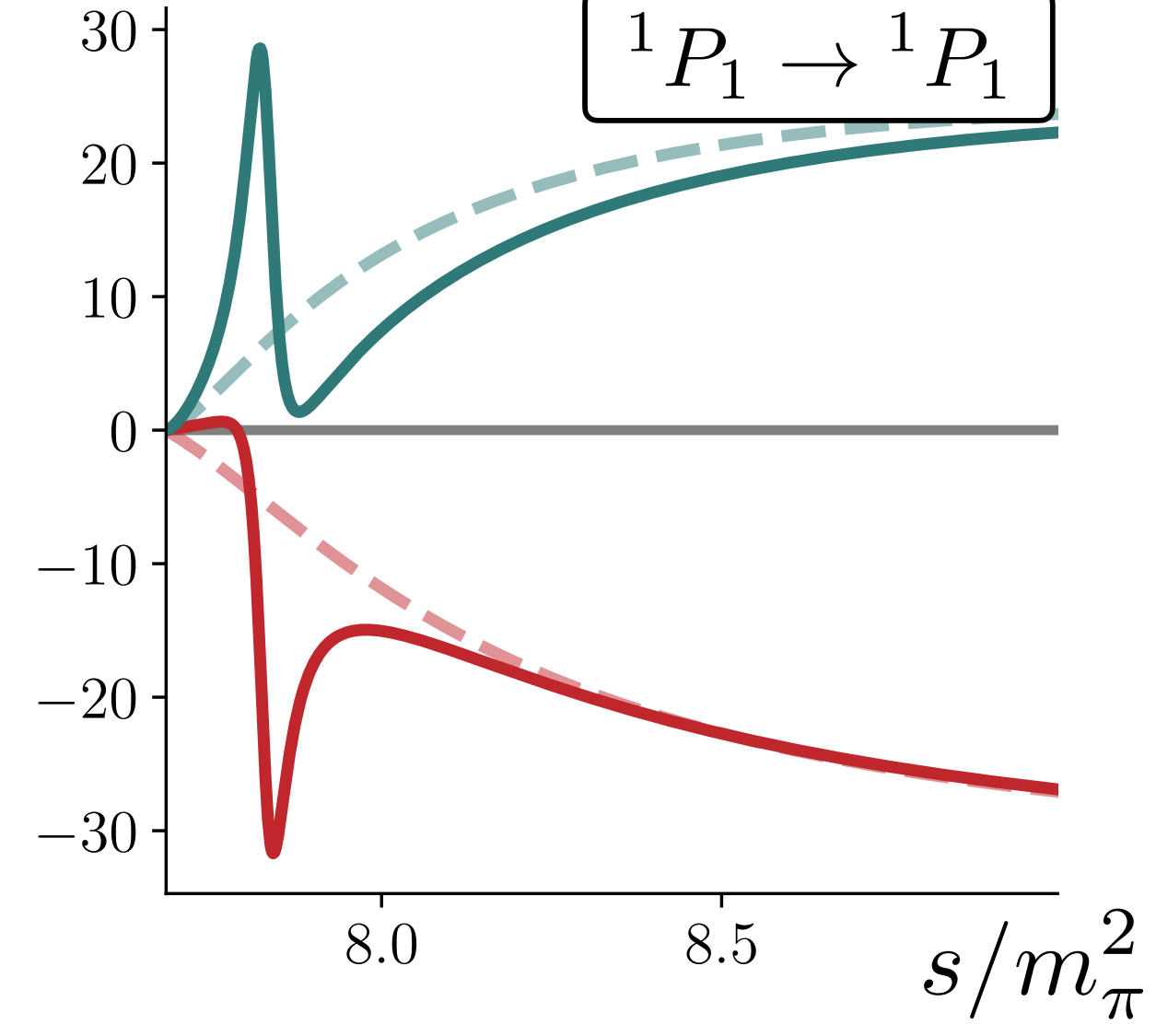
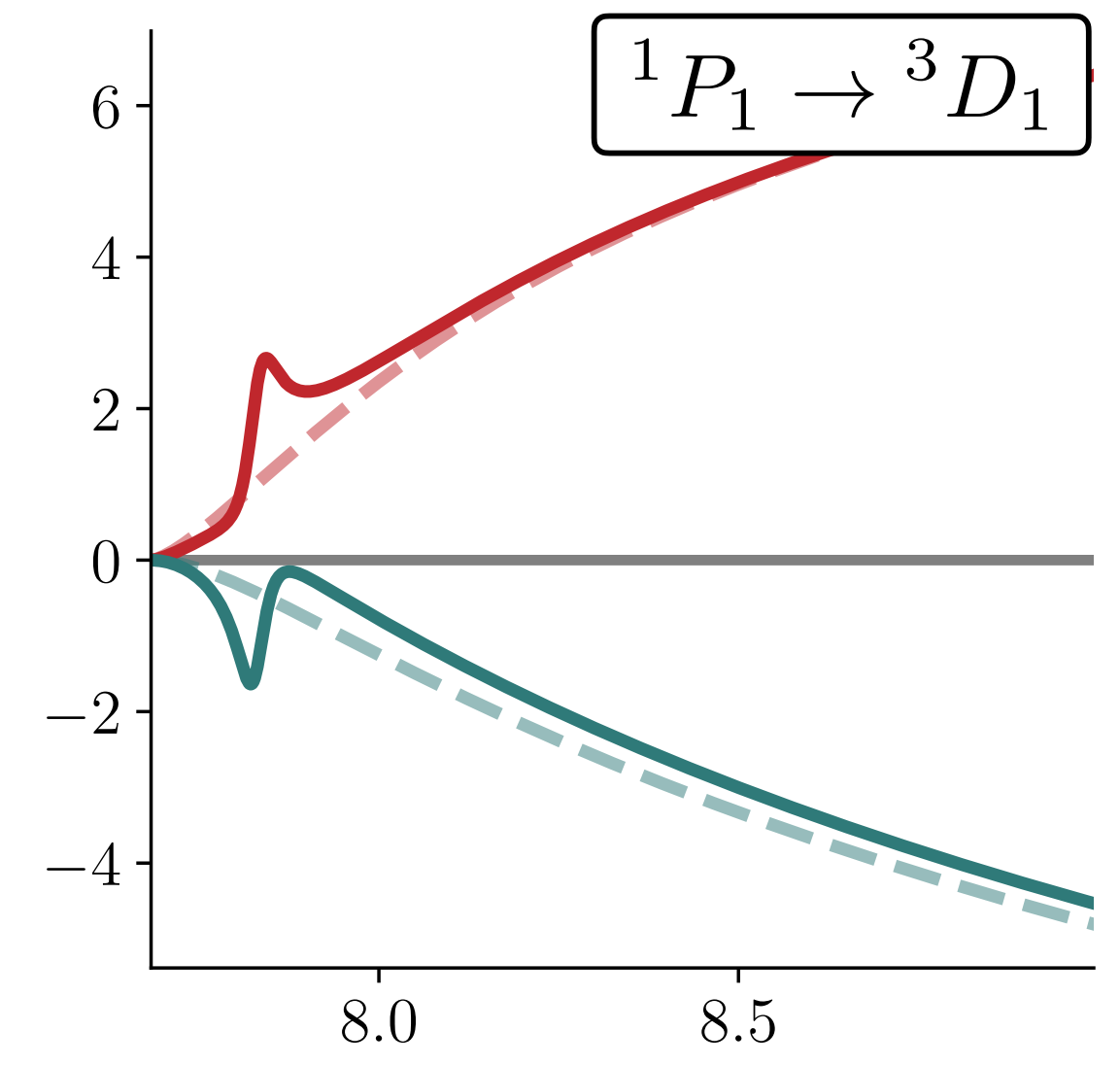
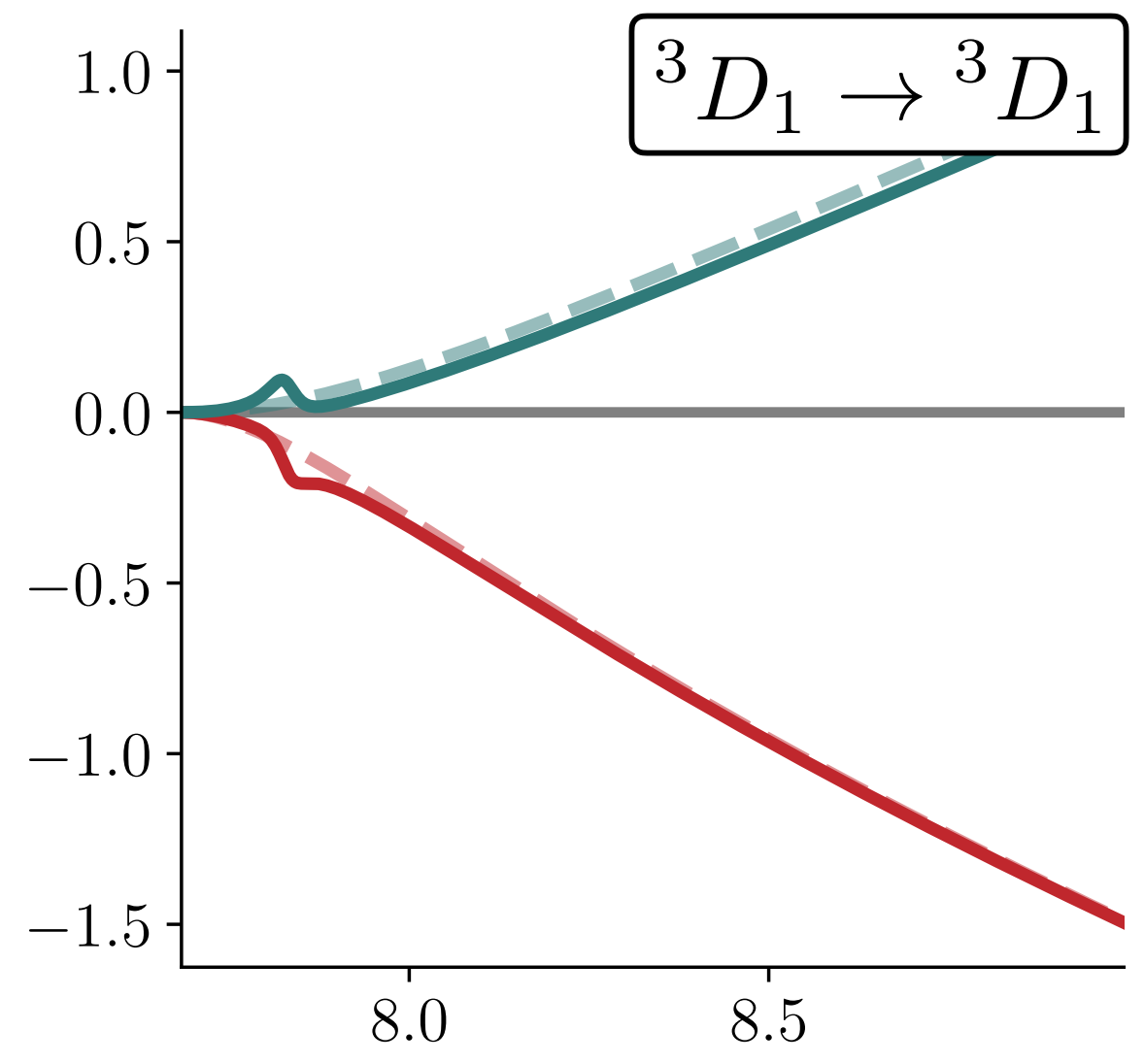
$$\mathcal{M}_{\varphi b}^{1(1^+)} = \begin{pmatrix} \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3S_1) & \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3D_1) & \mathcal{M}_{\rho\pi,\sigma\pi}({}^3S_1|{}^1P_1) \\ & \mathcal{M}_{\rho\pi,\rho\pi}({}^3D_1|{}^3D_1) & \mathcal{M}_{\rho\pi,\sigma\pi}({}^3D_1|{}^1P_1) \\ & & \mathcal{M}_{\sigma\pi,\sigma\pi}({}^1P_1|{}^1P_1) \end{pmatrix}$$

$$\mathcal{M}(^{2S'+1}L'_1 | ^{2S+1}L_1)$$



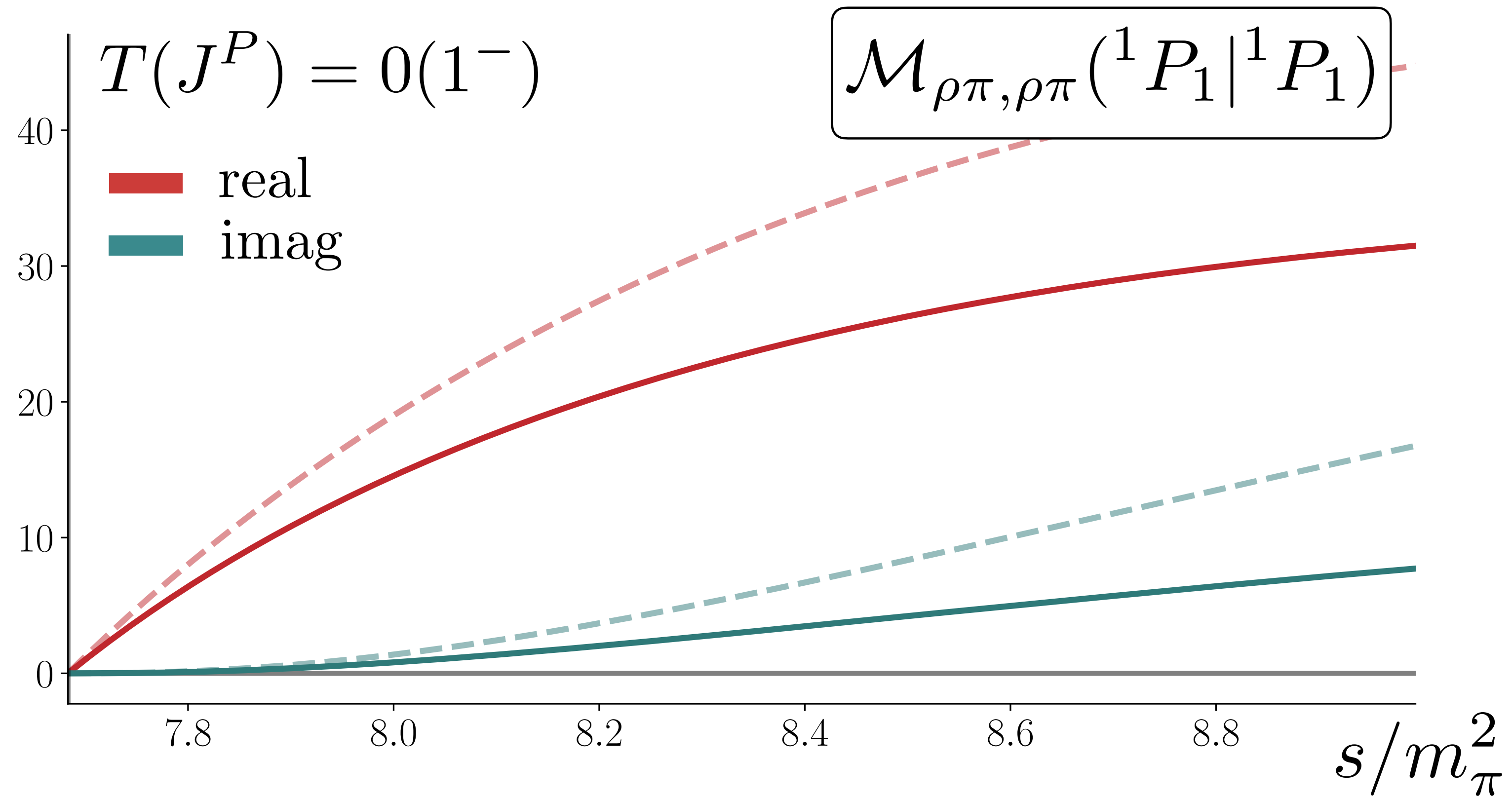
$$T(J^P) = 1(1^+)$$

■ real
■ imag



s/m_π^2

$T(J^P) = 0(1^-)$ channel with stable ρ



Summary & Perspectives

- Overview on 2- and 3-body scattering amplitudes
- Integral equations for PW projected 3-body amplitudes
- Factorizable K-matrices
- Toy model calculations for 3pi-systems

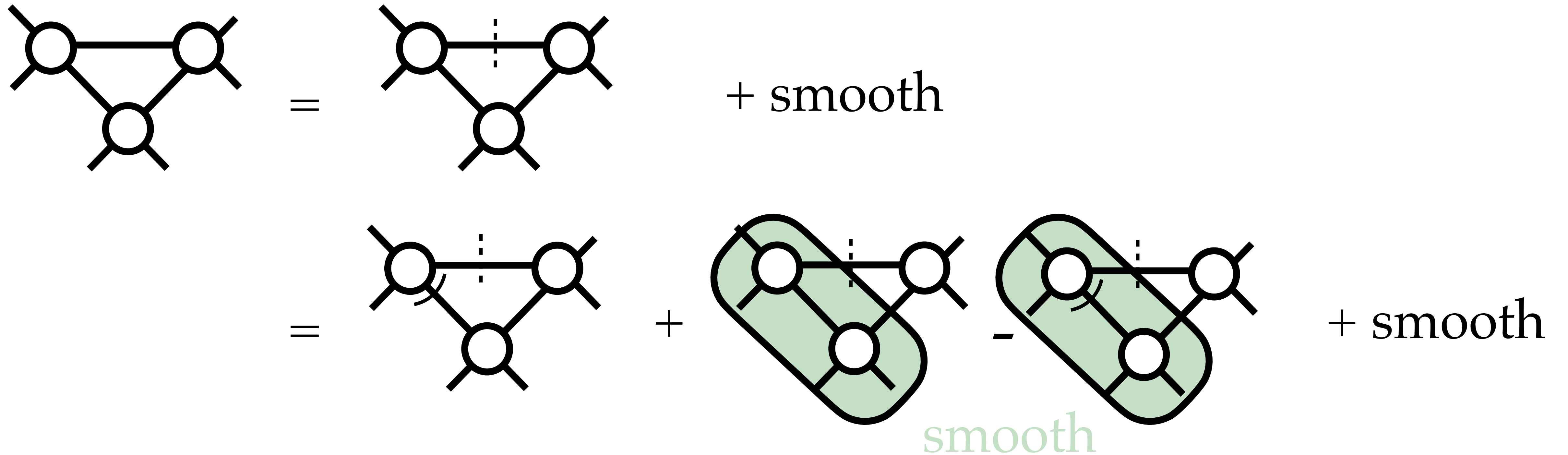
To do:

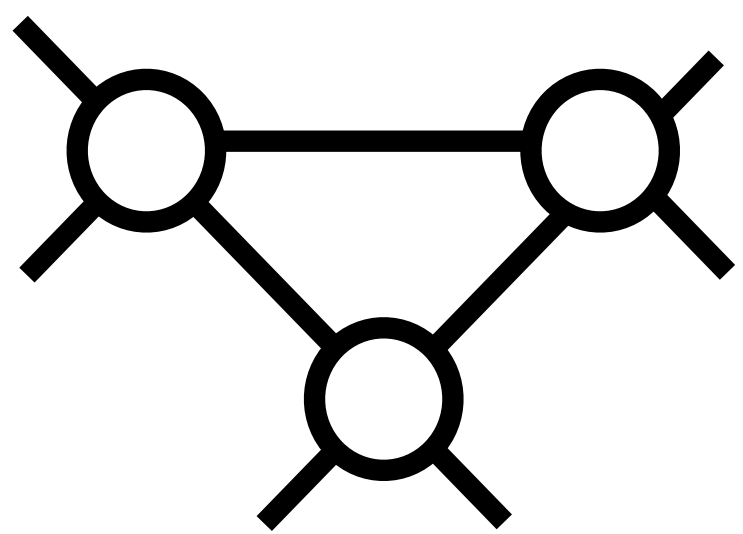
- Generalization to particles with spin, including 3N
- Checks on the finite-volume formalism (see Raúl's talk)

Backup

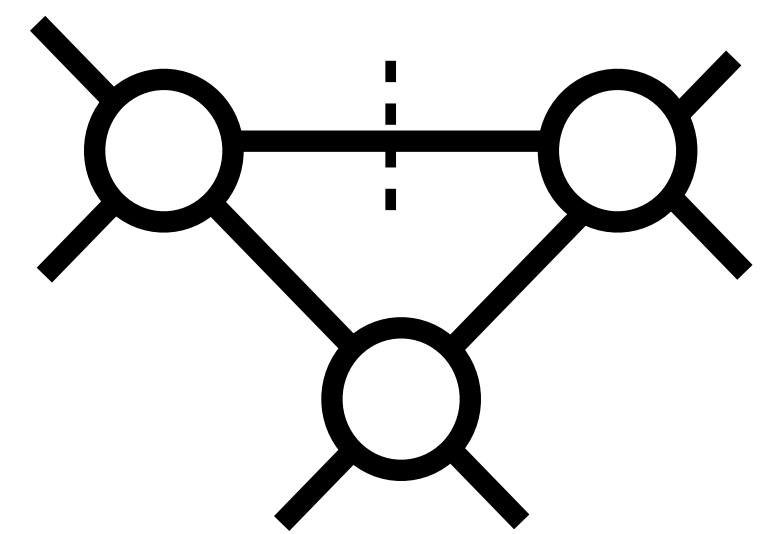
Reducing 4D to 3D

- ▣ Reducing from 4D to 3D while preserving singularities
- ▣ Remember, physical singularities are due to on-shell intermediate particles
- ▣ Let's consider a useful example:



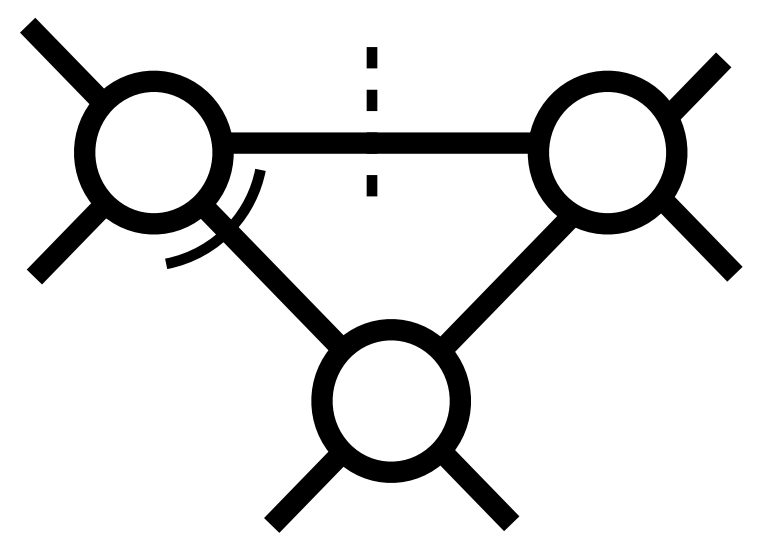


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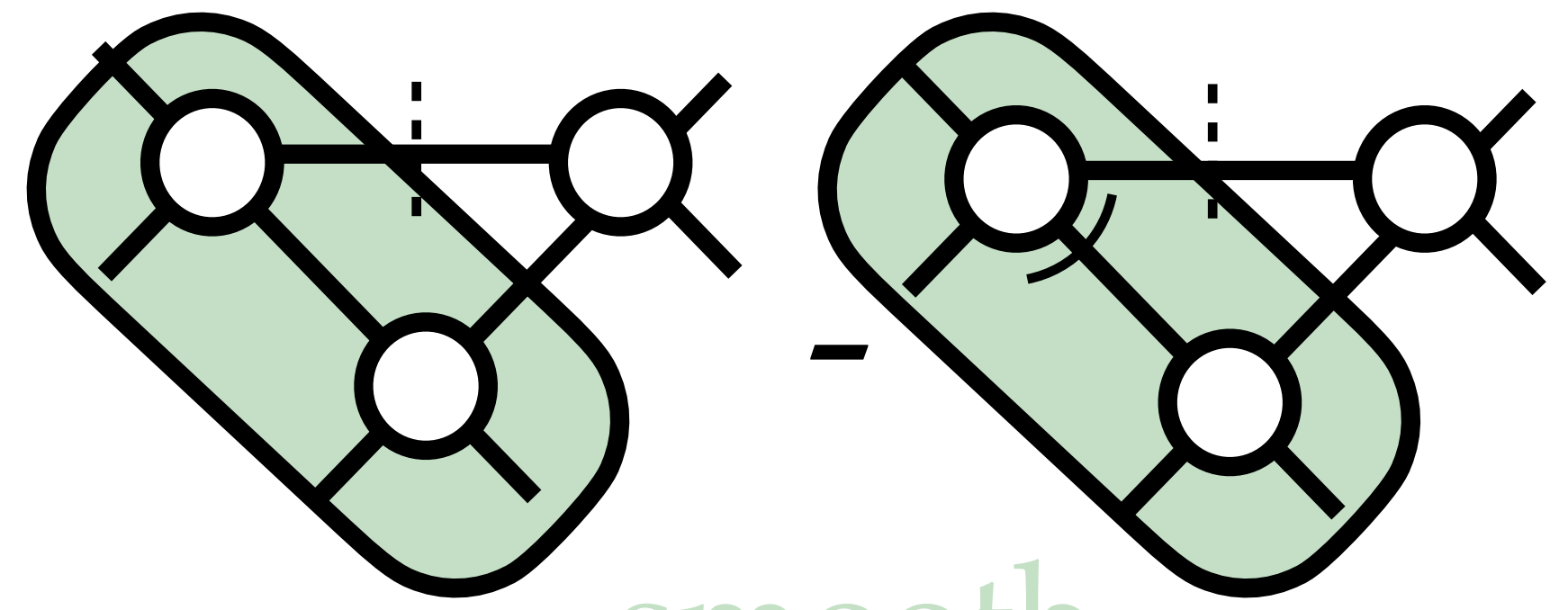


+ smooth

=



+



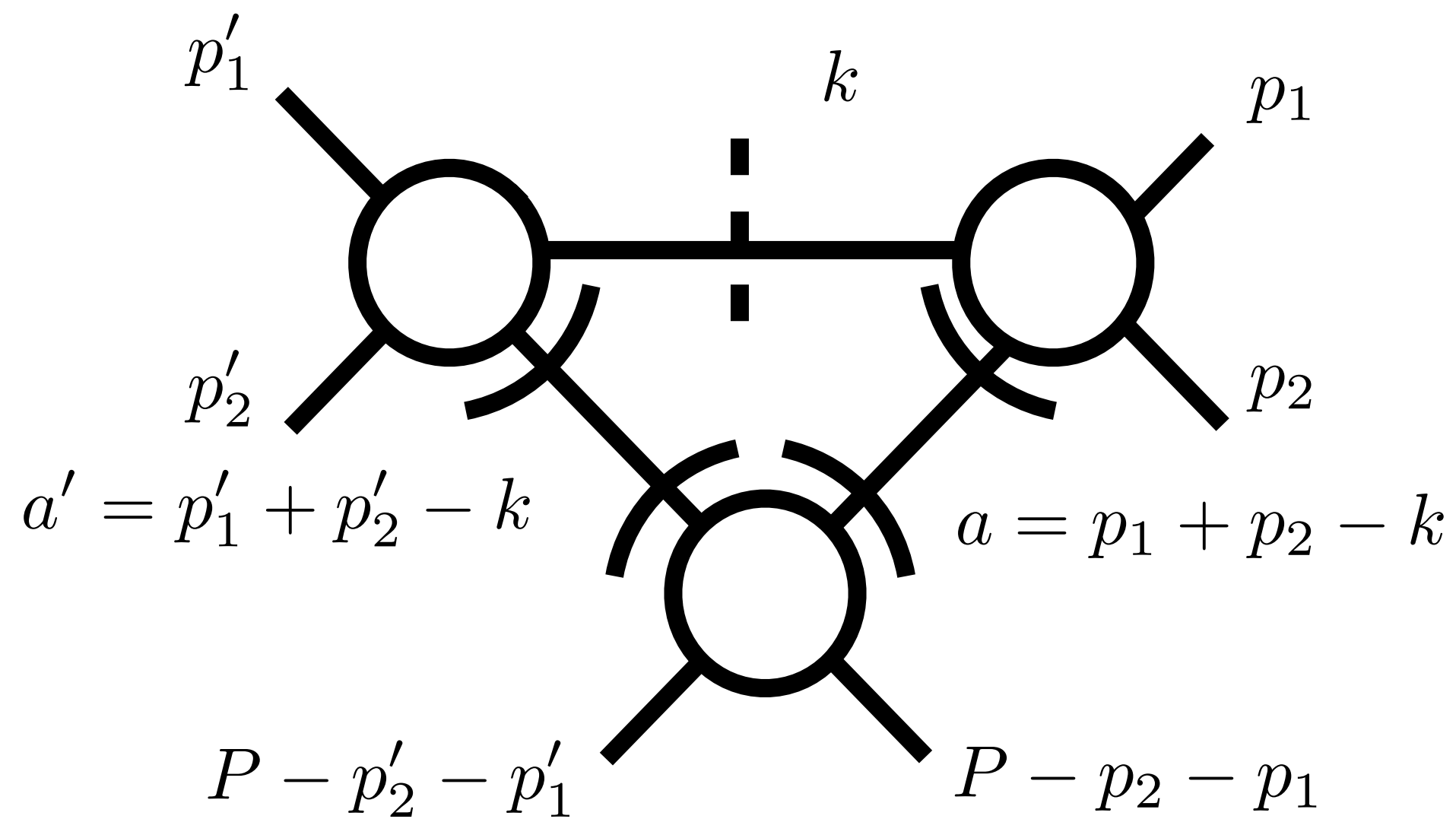
+ smooth

smooth

$$\sim \frac{B(p^2) - B(m^2)}{p^2 - m^2}$$

$$\approx \frac{B(m^2) + B'(m^2)(p^2 - m^2) - B(m^2)}{p^2 - m^2}$$

$$\approx B'(m^2)$$



$$= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} iB_{\text{on}}((p'_1 + p'_2)^2, \hat{k} \cdot \hat{p}'_1) \frac{i}{a'^2 - m^2} iB_{\text{on}}((P - k)^2, \hat{a} \cdot \hat{a}') \frac{i}{a^2 - m^2} iB_{\text{on}}((p_1 + p_2)^2, \hat{k} \cdot \hat{p}_1)$$

OPE

$$\mathcal{G}_{\ell'\lambda',\ell\lambda}(\mathbf{p}, \mathbf{k}) \equiv \frac{\mathcal{H}_{\lambda'\lambda}^{(\ell'\ell)}(\mathbf{p}, \mathbf{k})}{u_{pk} - m_e^2 + i\epsilon} ;$$

$$\mathcal{H}_{\lambda'\lambda}^{(\ell'\ell)}(\mathbf{p}, \mathbf{k}) \equiv \left(\frac{k_p^*}{q_p^*} \right)^{\ell'} 4\pi Y_{\ell'\lambda'}^*(\hat{\mathbf{k}}_p^*) Y_{\ell\lambda}(\hat{\mathbf{p}}_k^*) \left(\frac{p_k^*}{q_k^*} \right)^{\ell} .$$