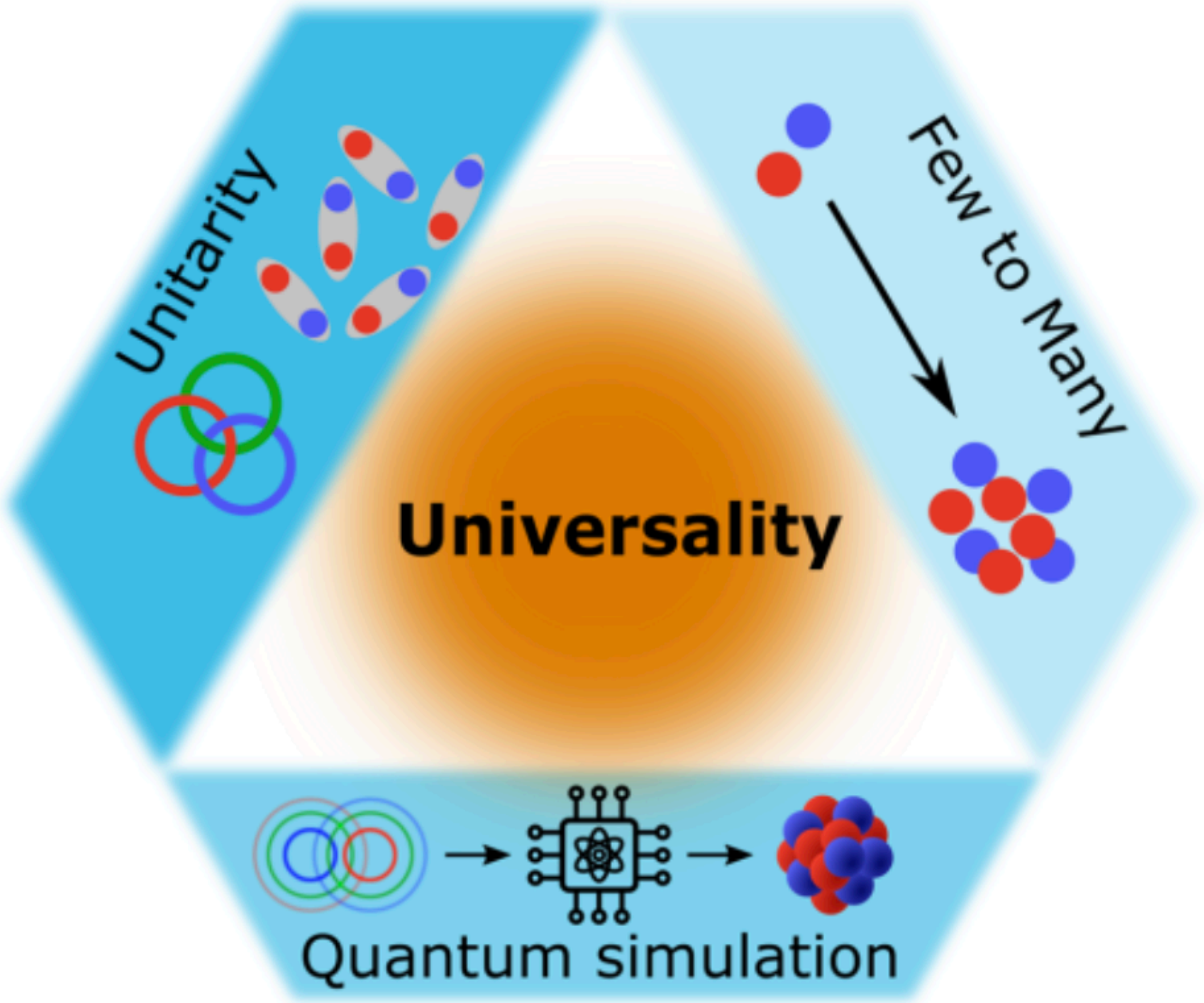
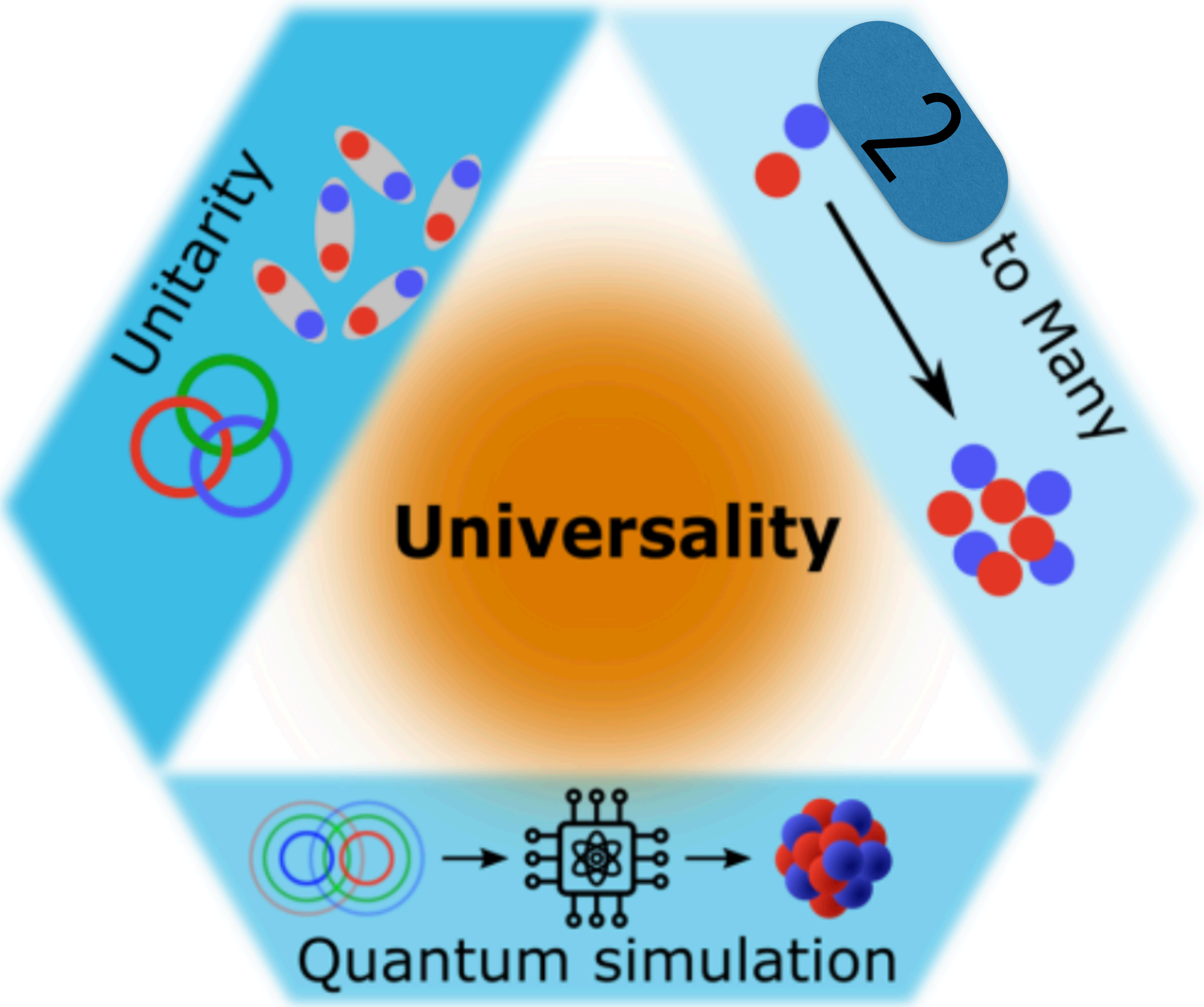


Caroline S. R. Costa

Quantum Few- and Many-Body Systems in Universal Regimes

October 7, 2024 - November 8, 2024

BERKELEY LAB



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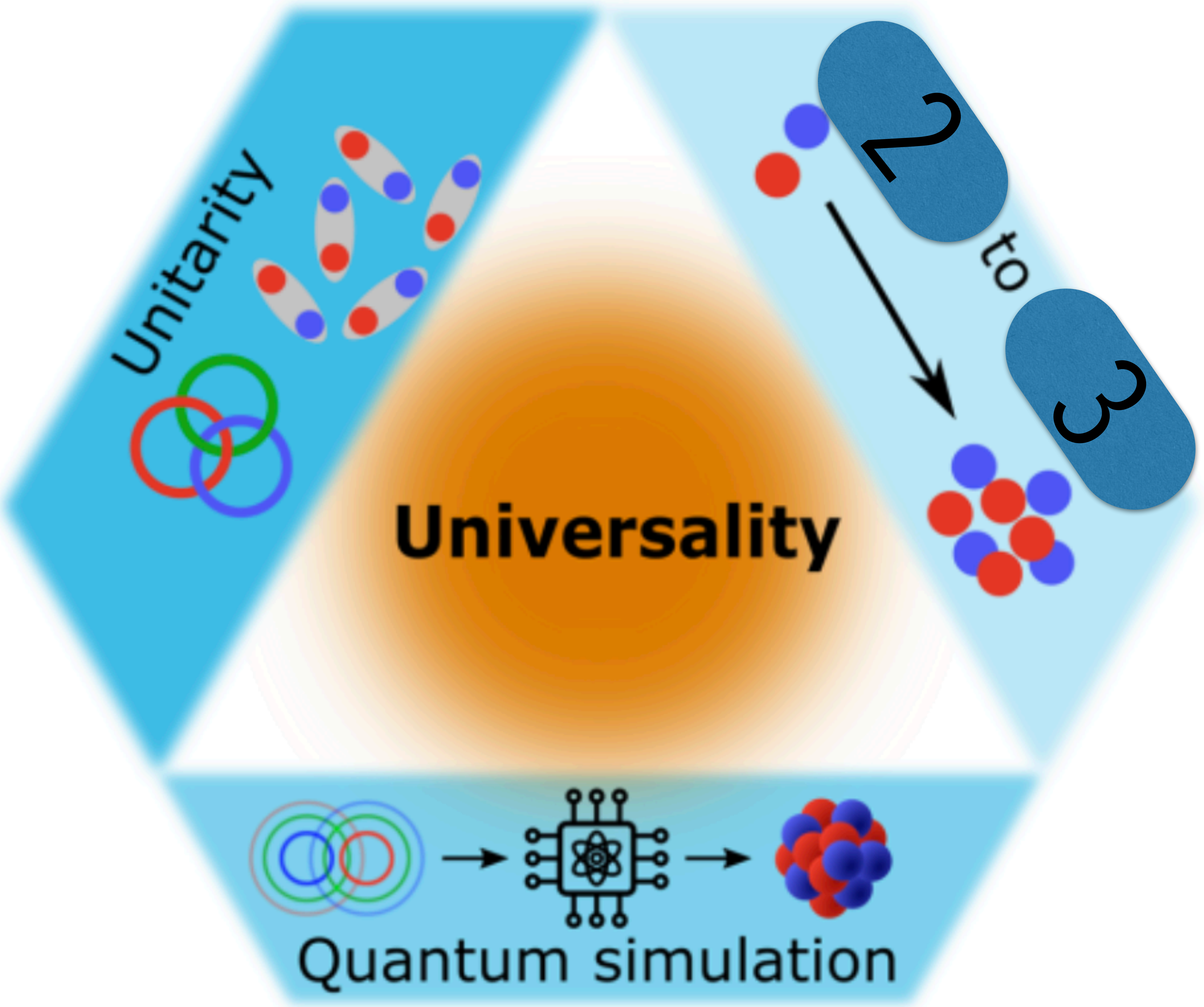

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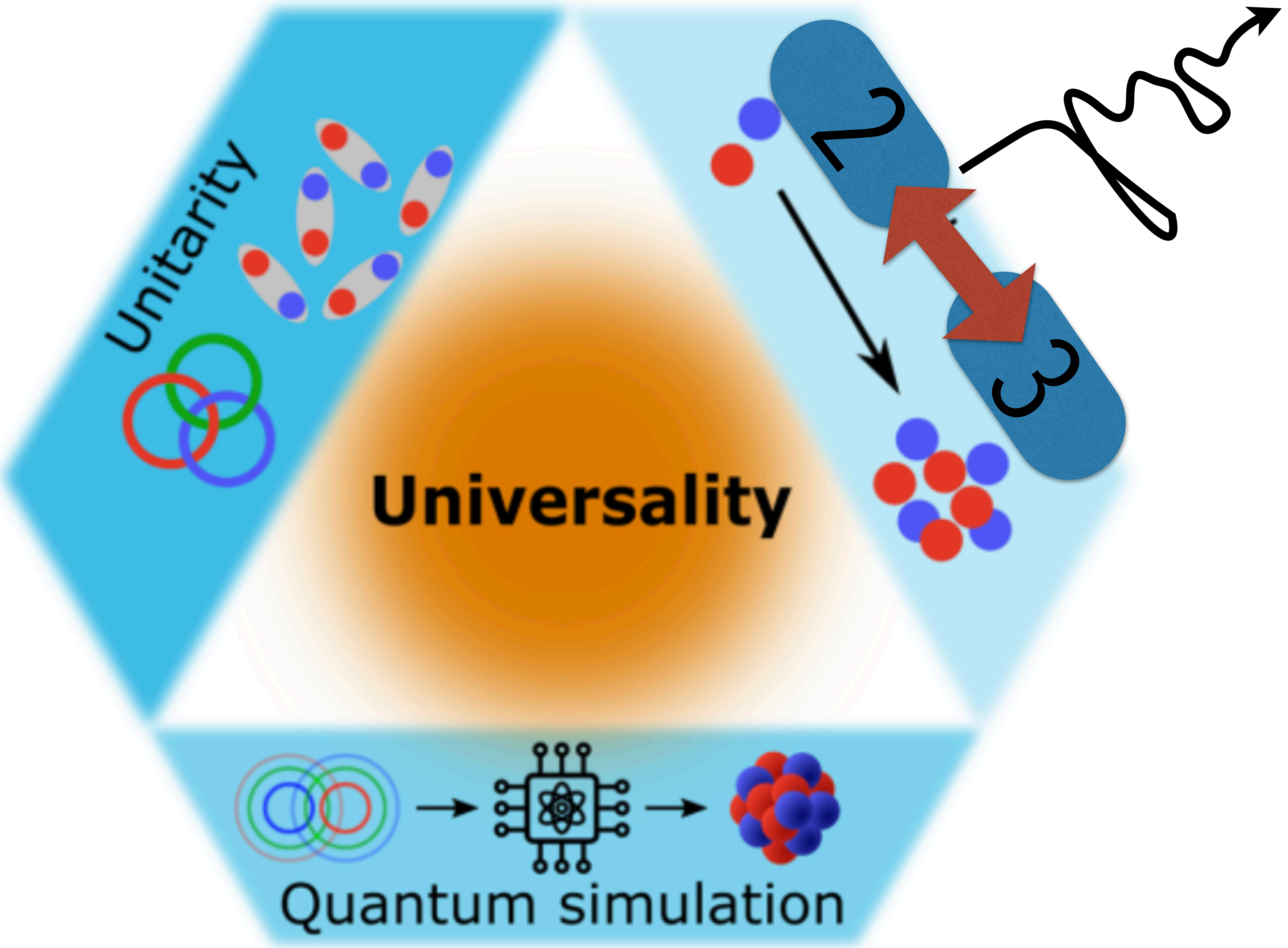

Hadrons systems

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Quantum Few- and Many-Body Systems in Universal Regimes

October 7, 2024 - November 8, 2024

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- 3N forces play a crucial role in describing nuclear properties
- Even at the 2-body sector, there are tensions ➡ 2n scattering length
- Traditionally, nuclear properties and nuclei formation have been described using phenomenological models and / or EFTs

A New Class of Three Nucleon Forces and their Implications

Vincenzo Cirigliano,[1,*] Maria Dawid,[1,†] Wouter Dekens,[1,‡] and Sanjay Reddy[1,§]

[1] *Institute for Nuclear Theory, University of Washington, Seattle, WA 98195, USA*

(Dated: November 4, 2024)

We identify a new class of three-nucleon forces that arises in the low-energy effective theory of nuclear interactions including pions. We estimate their contribution to the energy of neutron and nuclear matter and find that it can be as important as the leading-order three-nucleon forces previously considered in the literature. The magnitude of this force is set by the strength of the coupling of pions to two nucleons and is presently not well constrained by experiments. The implications for nuclei, nuclear matter, and the equation of state of neutron matter are briefly discussed.

- Motivation: Constrain two- and three-body forces directly from Standard Model

Crucial also for probing BSM physics, e.g., in $0\nu\beta\beta$

outline

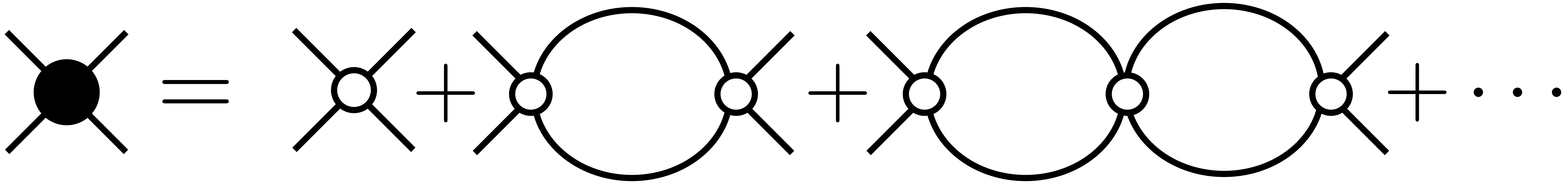
- ☐ relativistic integral equations [4D → 3D]
- ☐ angular momentum projection of OPE
- ☐ angular momentum projection of amplitudes
- ☐ LSZ for 3Body → 2Body
- ☐ toy models for 3pi [including isospin in the OPE]
- ☐ Numerical solutions // Unitary check

Hansen, RB, Edwards, Thomas, & Wilson (2020)
Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)
Jackura, RB (2023)
RB, S. R. Costa, Jackura, (2024)
Dawid, RB, Islam, Jackura, (2023)

Two-hadron systems

- Sum over all **2 → 2** amputated diagrams

$$i\mathcal{M}_2 =$$



Two-hadron systems

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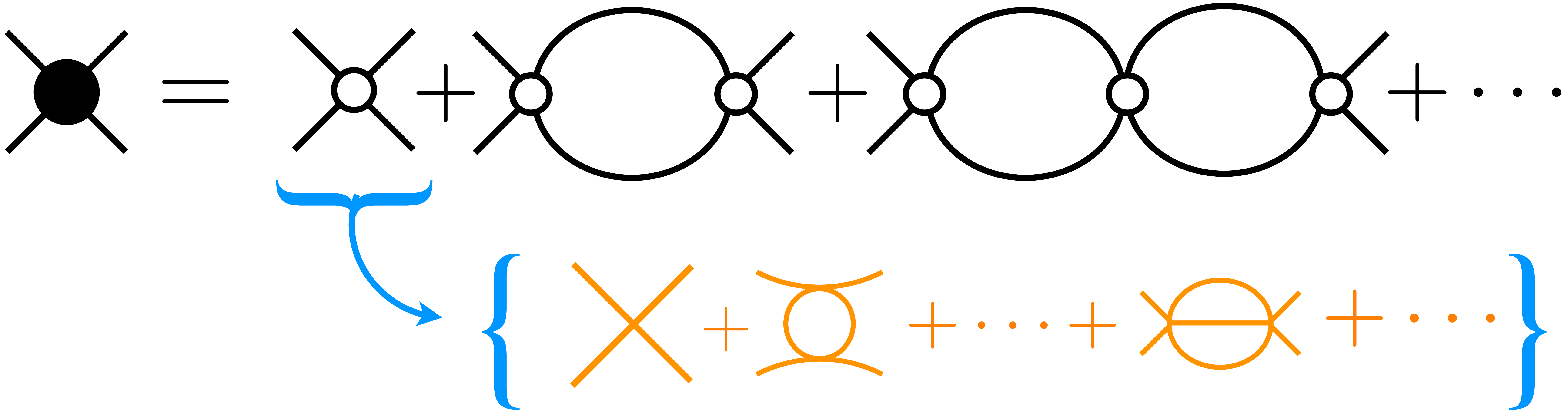
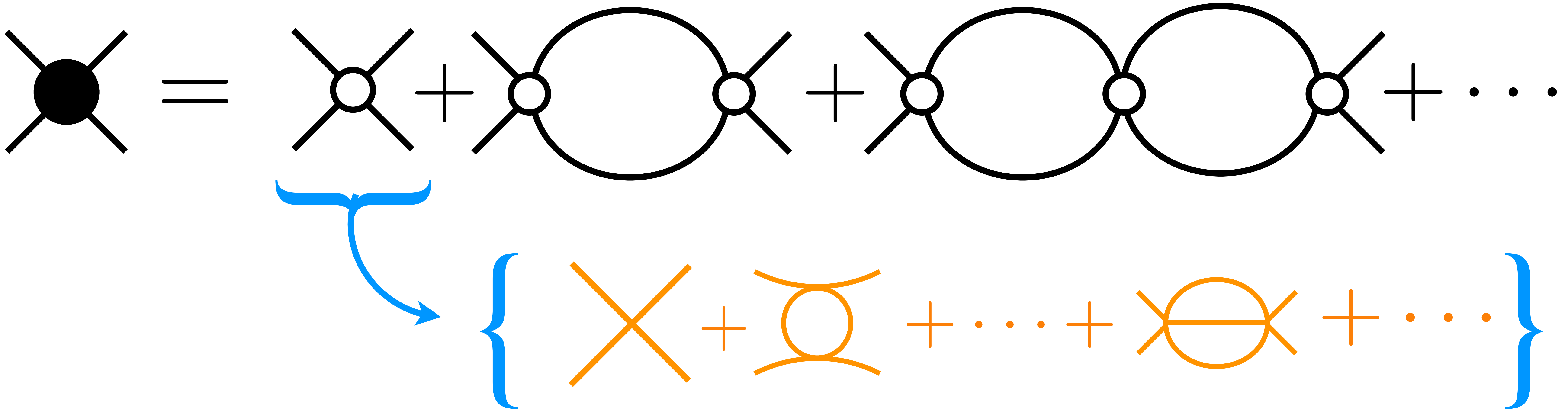


All 2-PI s-channel diagrams

Two-hadron systems

- Sum over all **2 ➔ 2** amputated diagrams

$$i\mathcal{M}_2 =$$

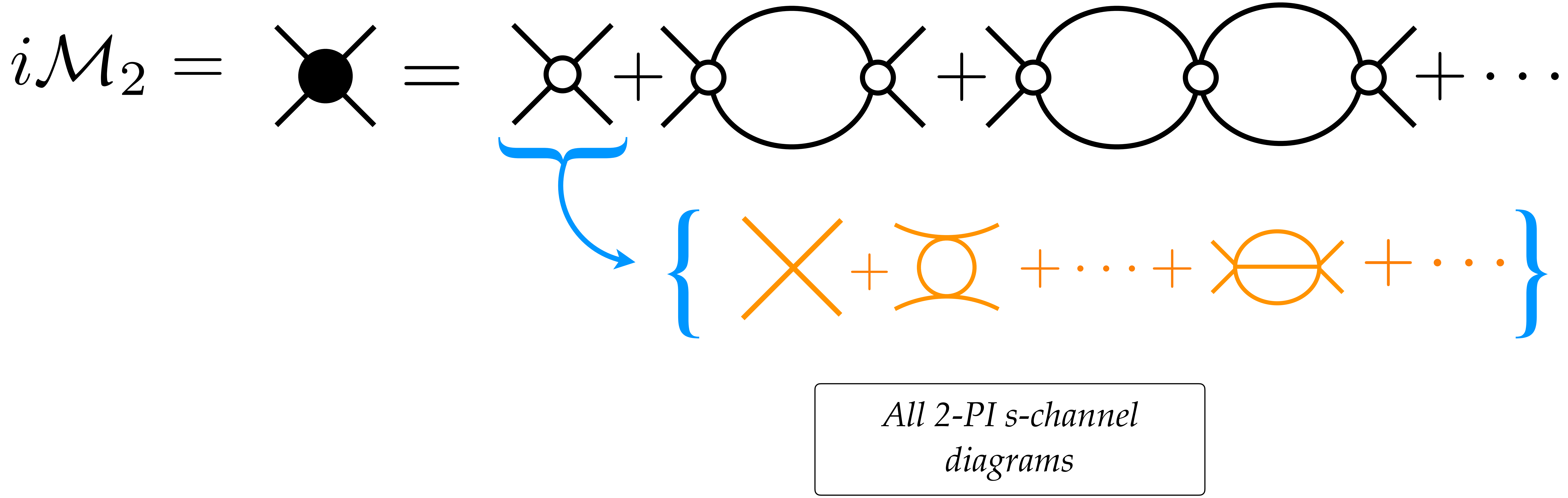


All 2-PI s-channel diagrams

- The goal: Isolate all the singularities of the scattering amplitude!

Two-hadron systems

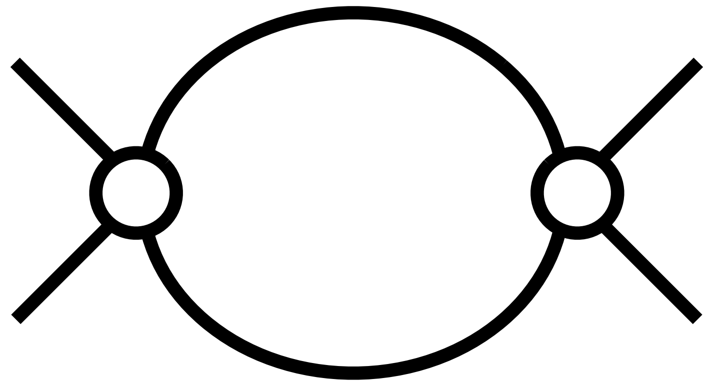
- Sum over all **2 ➔ 2** amputated diagrams



- The goal: Isolate all the singularities of the scattering amplitude!
- Kernel is not singular in the kinematic region of interest
- Singularities are due to intermediate particles going on-shell

Two-hadron systems

- The goal: Isolate singularities from two particle on shell states in bubble diagrams



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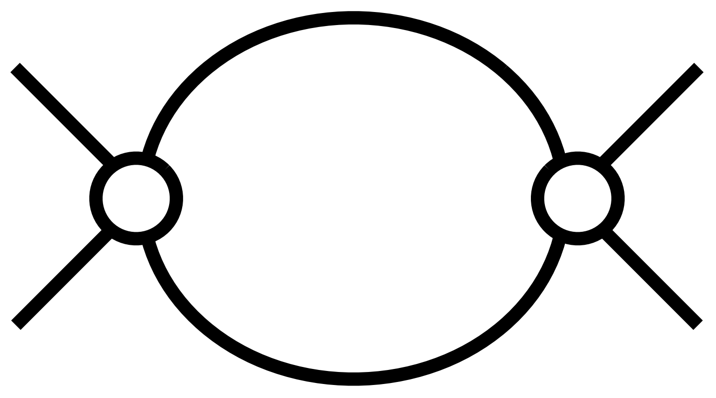
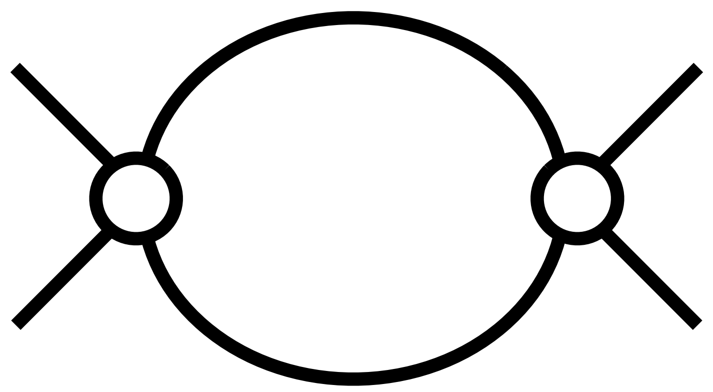
- The goal: Isolate singularities from two particle on shell states in bubble diagrams



$$= \int \frac{d^4k}{(2\pi)^4} [iB(k,P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P-k)^2 - m^2 + i\epsilon}$$

Two-hadron systems

- The goal: Isolate singularities from two particle on shell states in bubble diagrams



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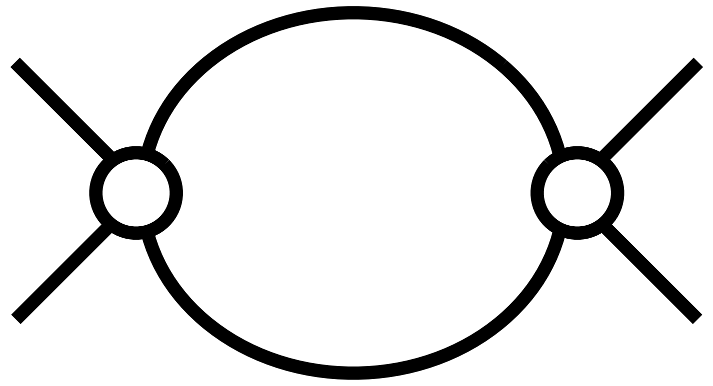
$$= \int \frac{dk_0}{2\pi} \frac{d^3\vec{k}}{(2\pi)^3}[iB(k,P)]^2 \frac{i}{k_0^2 - \omega_k^2 + i\epsilon} \frac{i}{[(k_0 - E)^2 - \omega_{kp}^2 + i\epsilon]}$$

Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams



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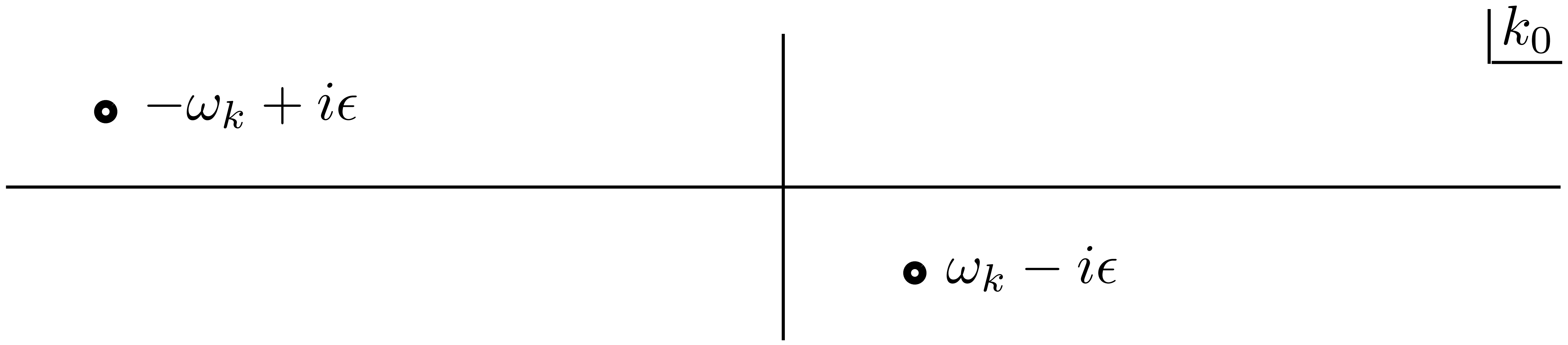
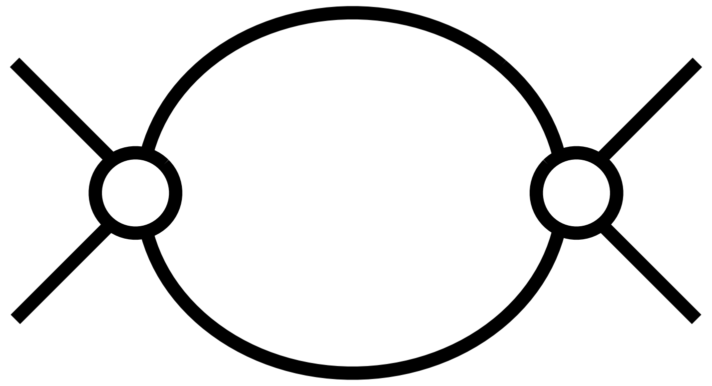


Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams



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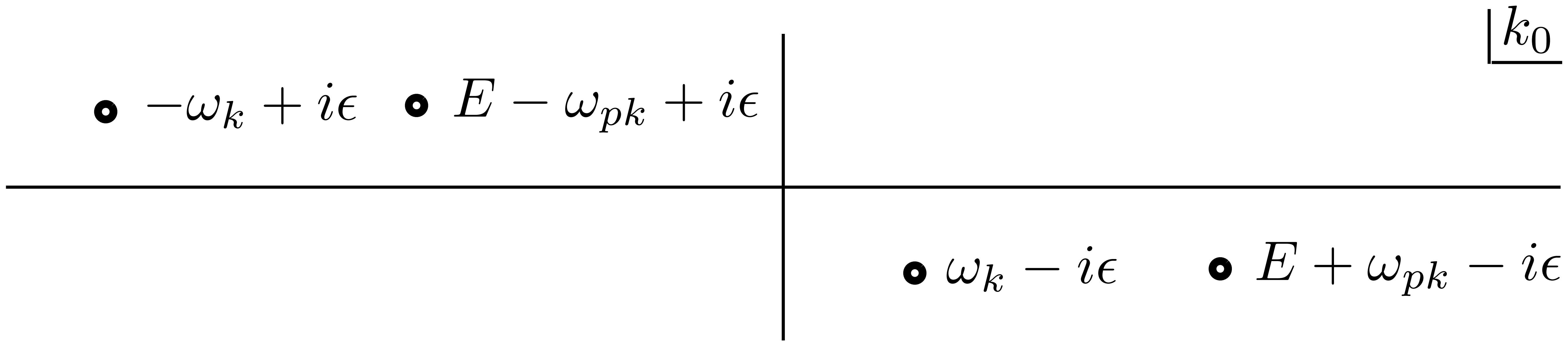
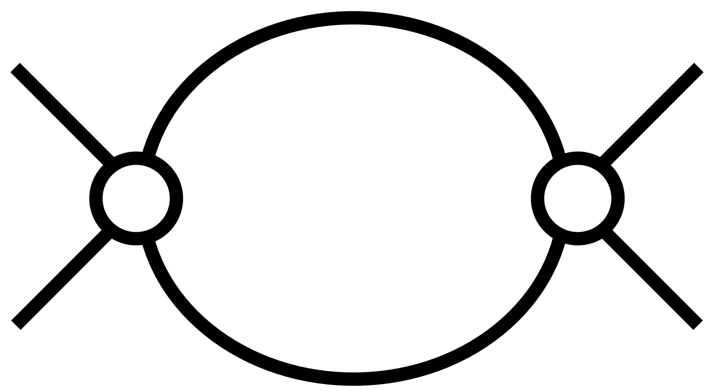


Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams



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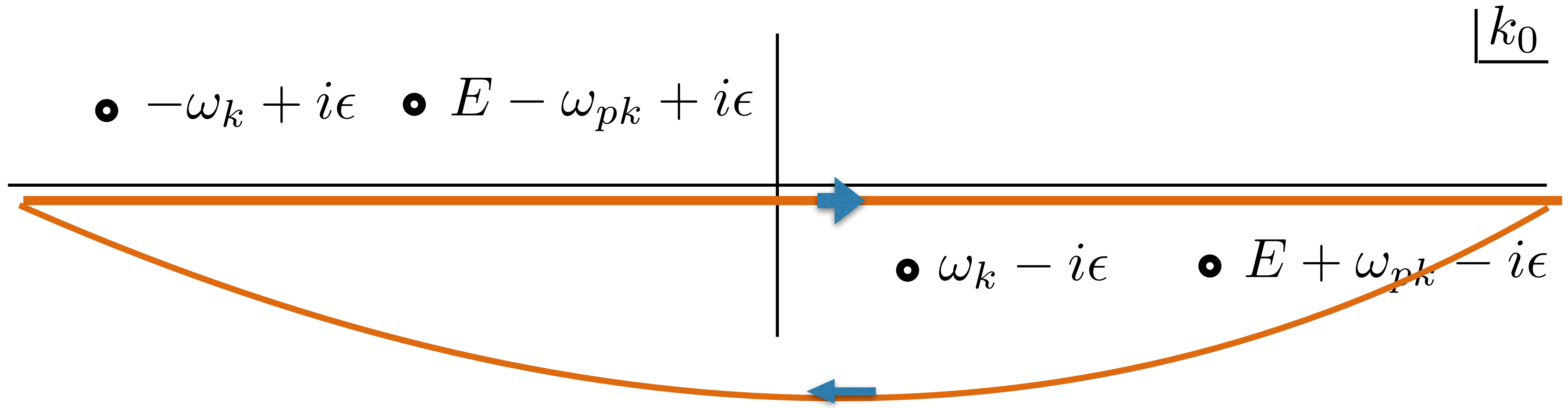
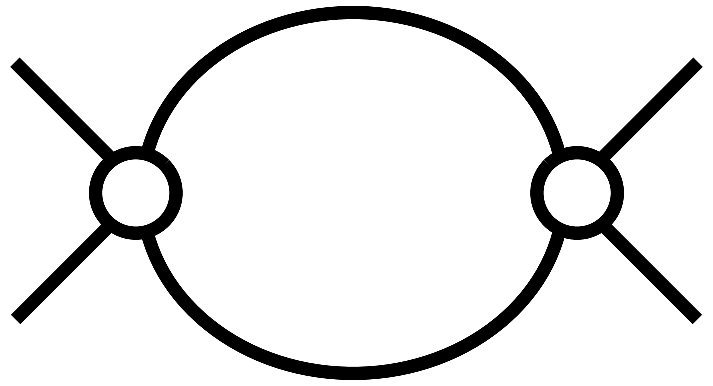
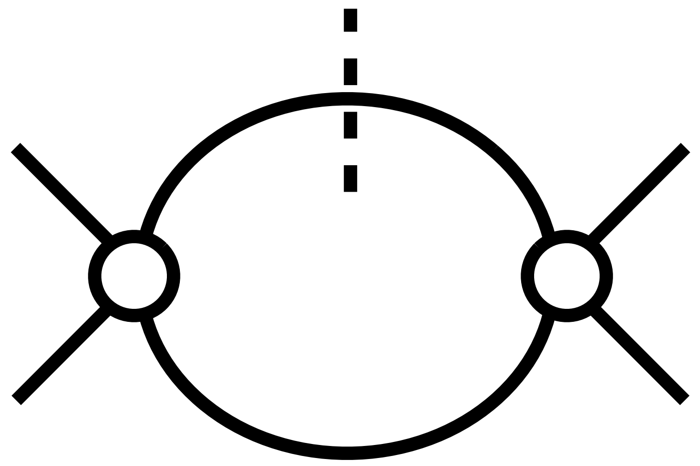


Two-hadron systems

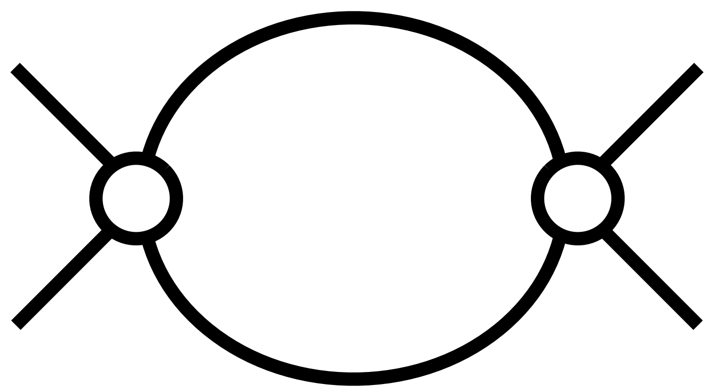
- Isolate singularities from two particle on shell states in bubble diagrams

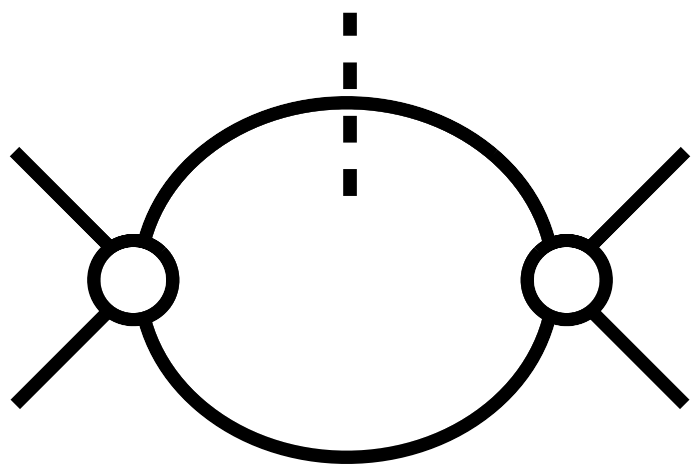


$=$  $+$ "smooth"

Two-hadron systems

- Isolate singularities from two particle on shell states in bubble diagrams

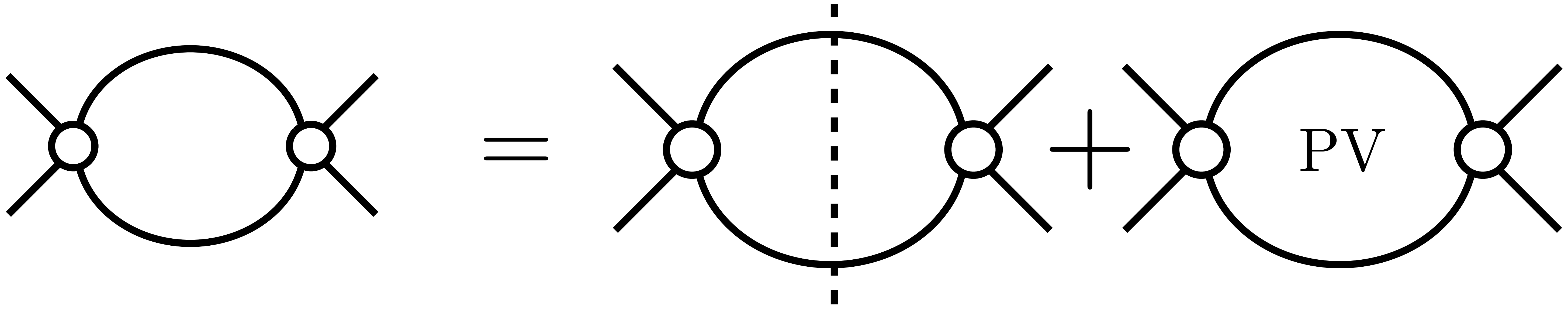


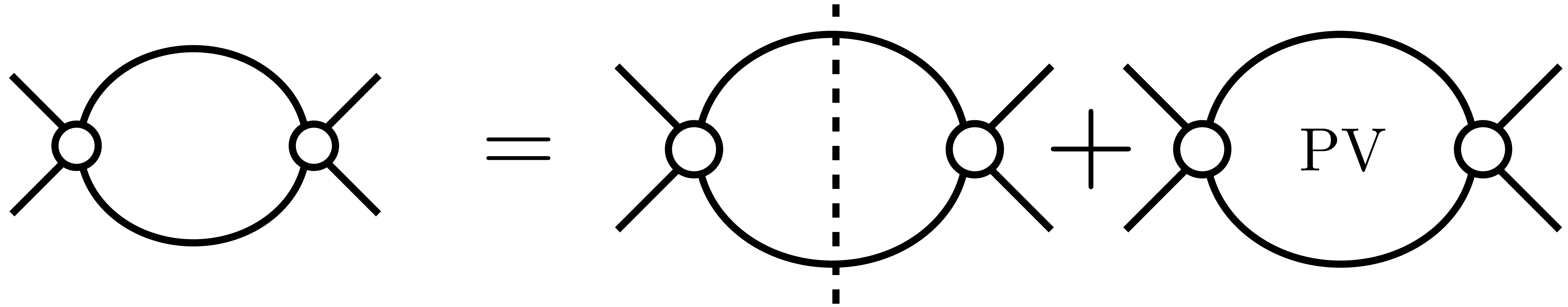
$=$  $+$ "smooth"

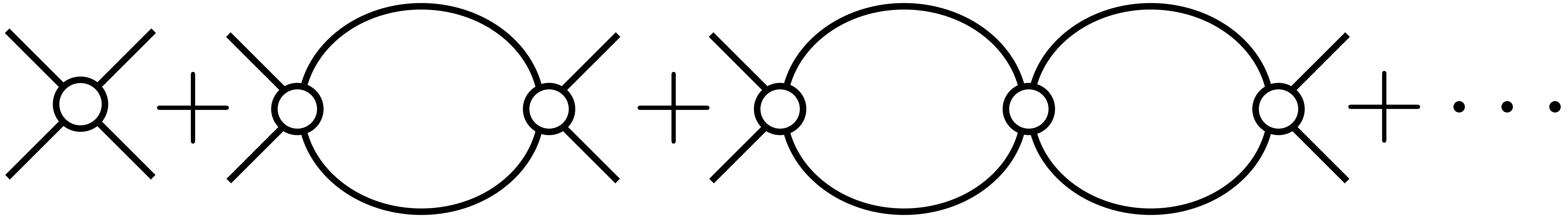
$$= \int \frac{d^3k}{(2\pi)^3} \frac{[iB(k,P)]^2}{(2\omega_k)^2} \pi\delta(E - 2\omega_k) + \text{“PV integral”}$$

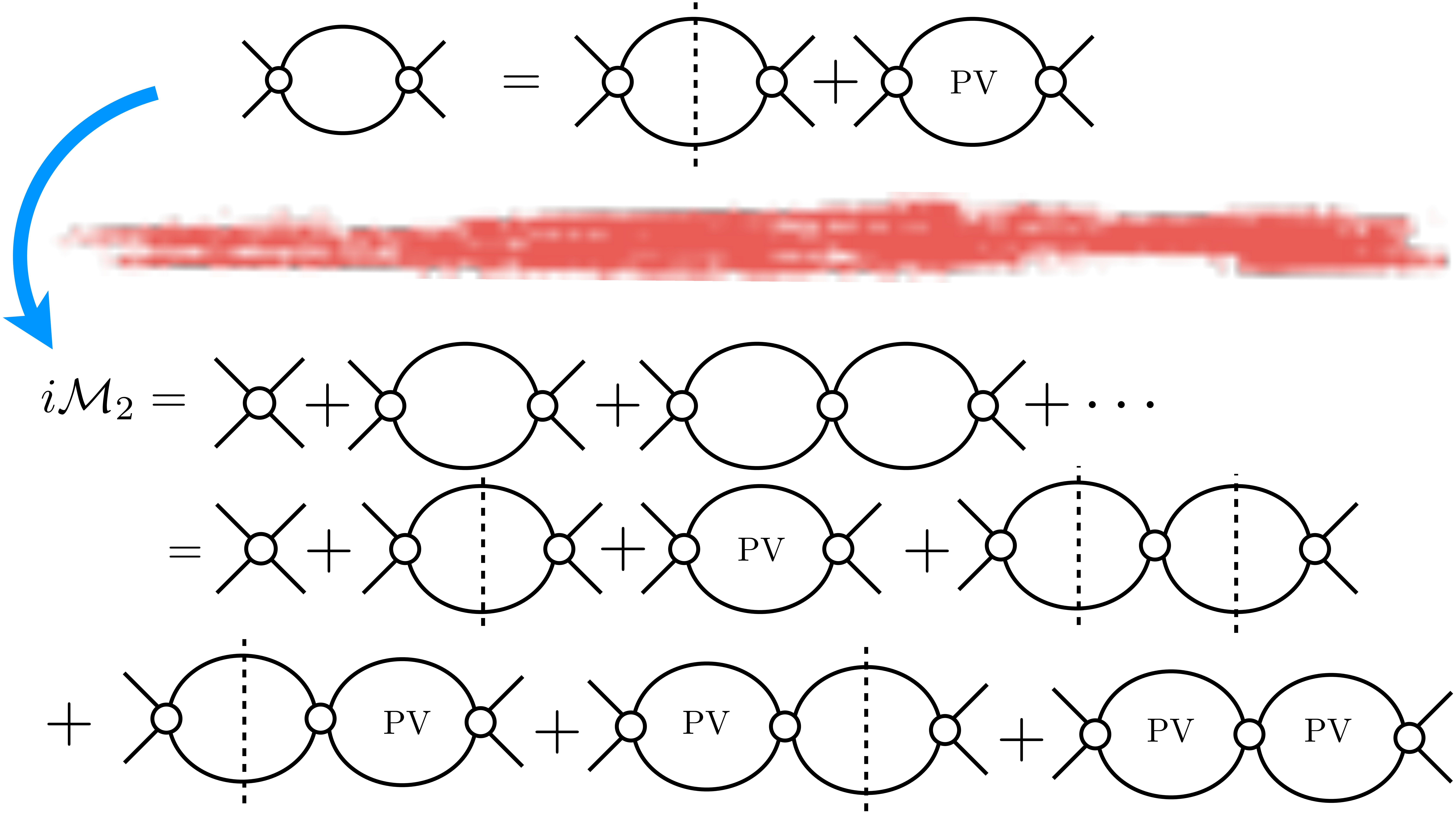
$$= [iB_{on}]\rho\,[iB_{on}] + \text{“PV integral”}$$

$$\rho \equiv \frac{p}{8\pi E} \sim \sqrt{s - s_{th}}$$ "Square root singularity"


PV

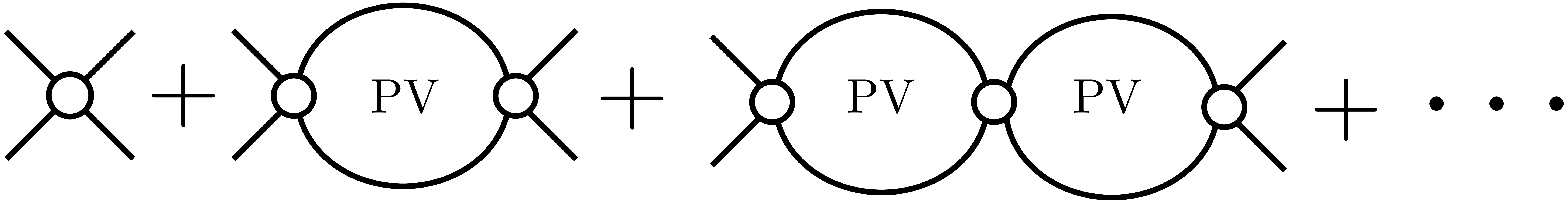


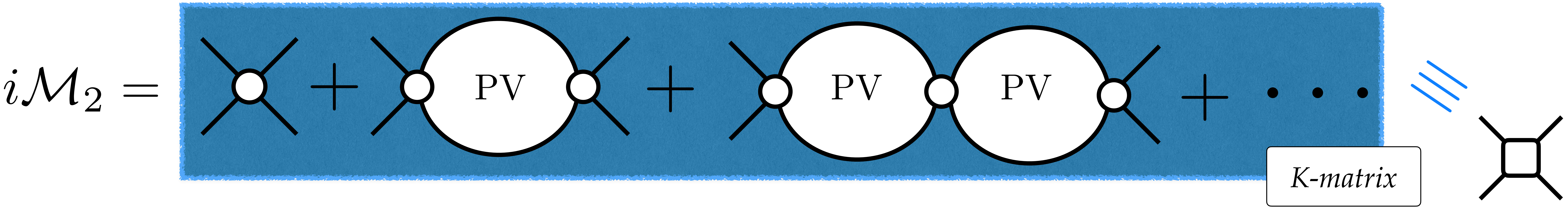
$i\mathcal{M}_2 =$ 

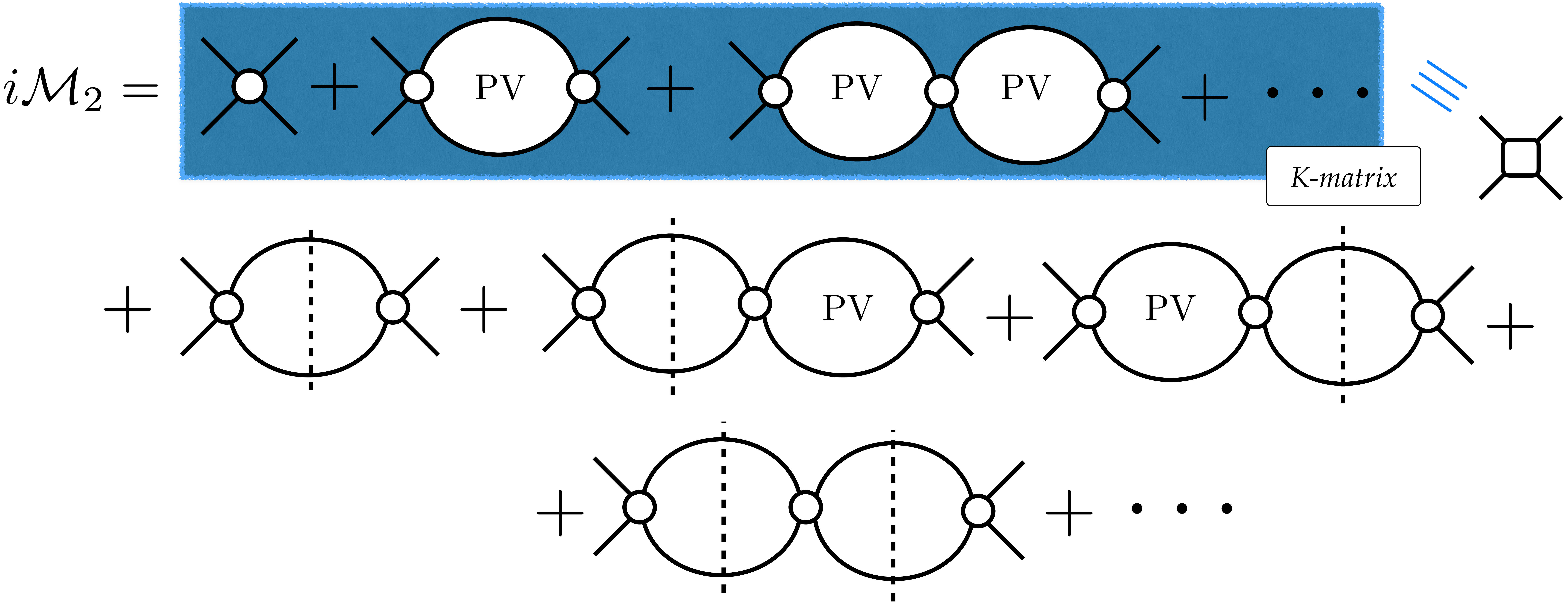


PV
iM₂ =
PV
PV
PV
PV
PV

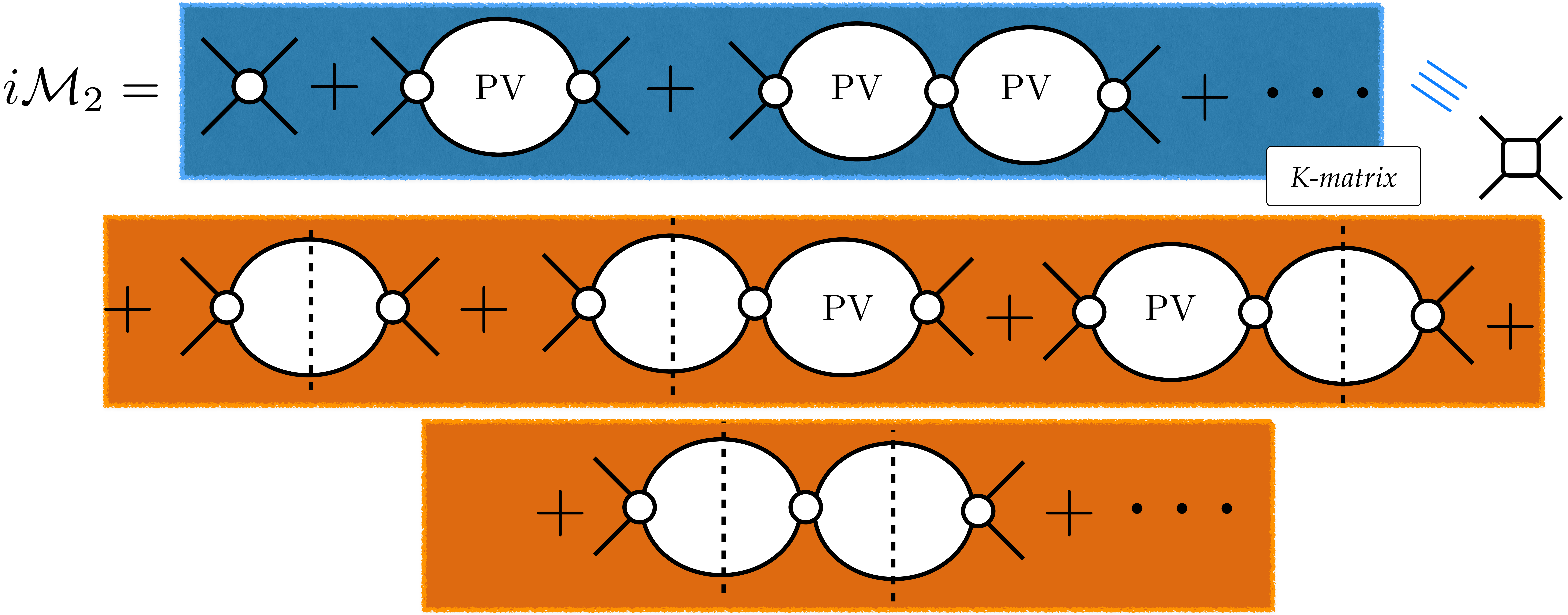
$$i\mathcal{M}_2 =$$




$i\mathcal{M}_2 =$
PV
PV
PV
K-matrix



$i\mathcal{M}_2 =$
PV
PV
PV
K-matrix
PV
PV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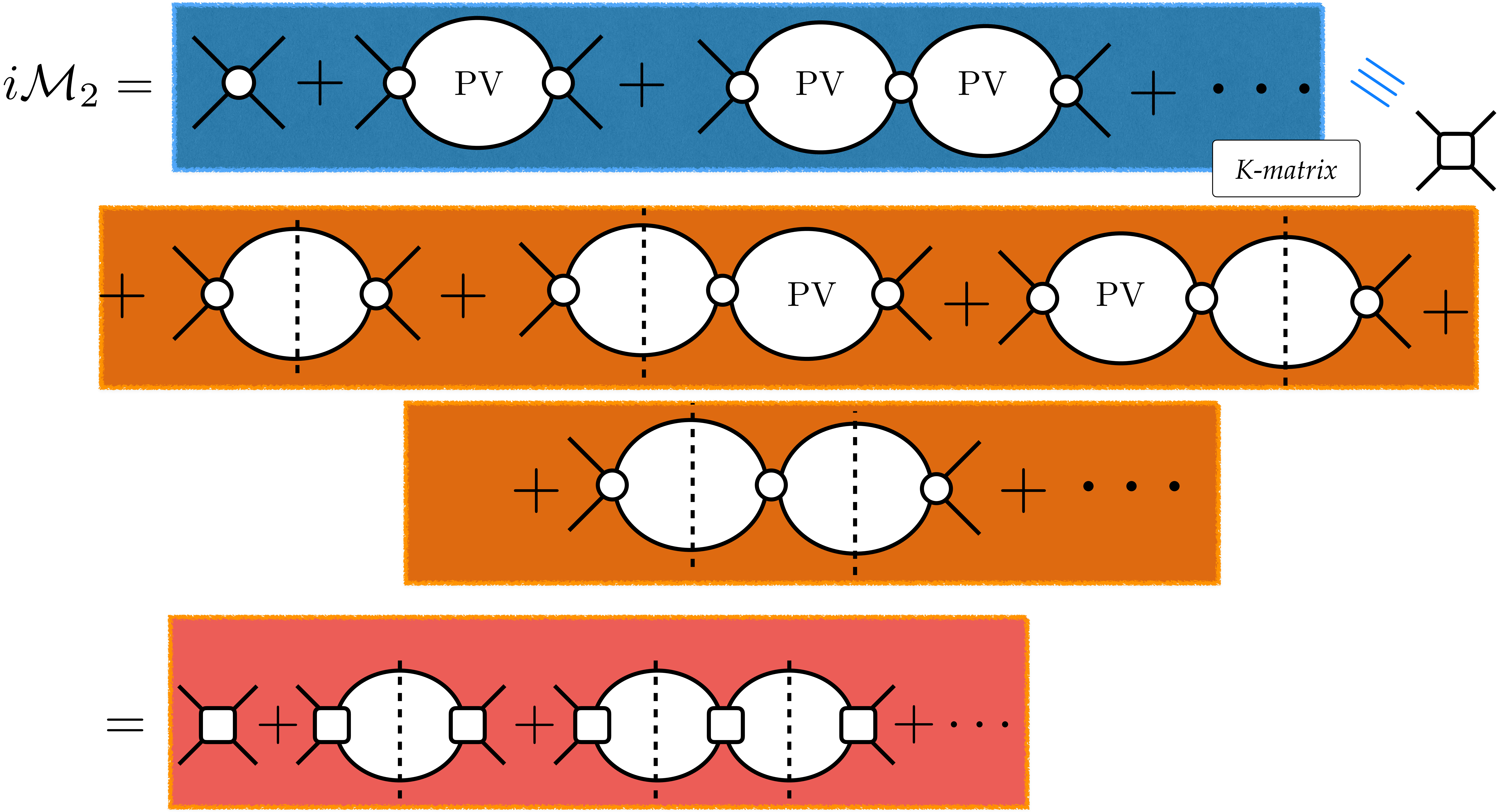
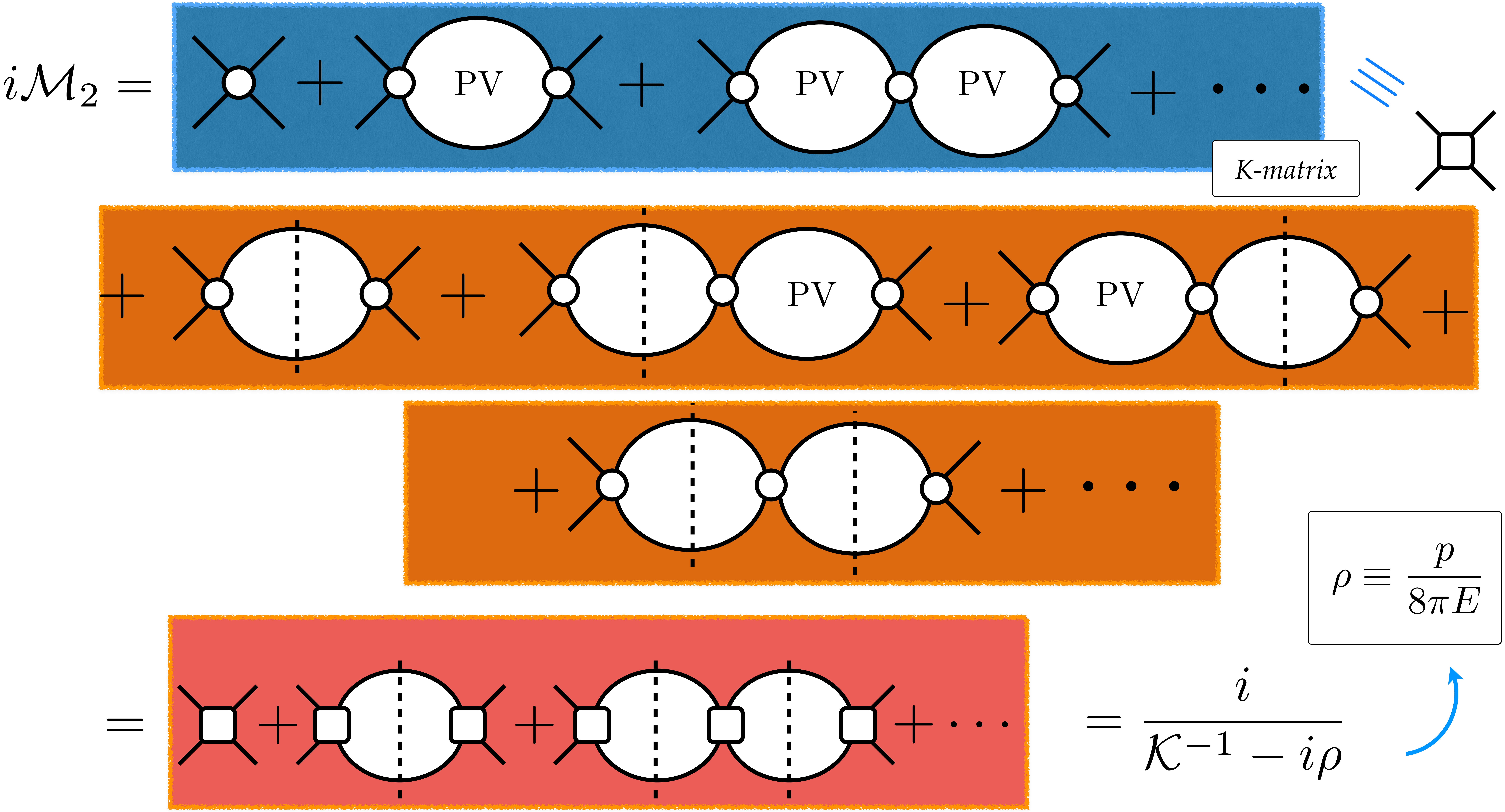
PV
PV
PV
K-matrix
PV
PV


$i\mathcal{M}_2 =$
PV
PV
PV
K-matrix
PV
PV
=

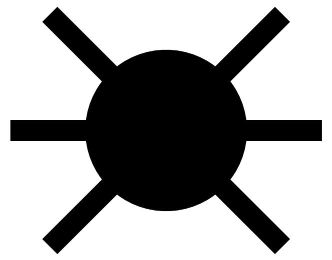
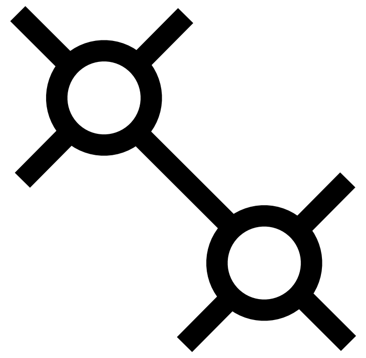
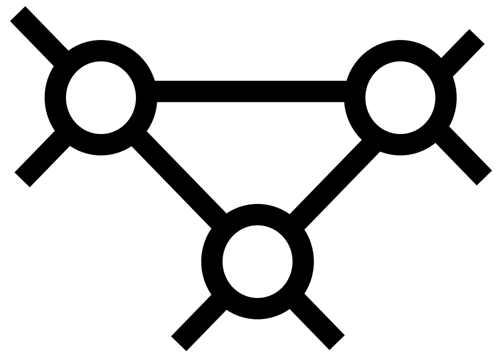


$i\mathcal{M}_2 =$
PV
+ . . .
K-matrix
$\rho \equiv \frac{p}{8\pi E}$
$= \frac{i}{\mathcal{K}^{-1} - i\rho}$

Three-hadron systems

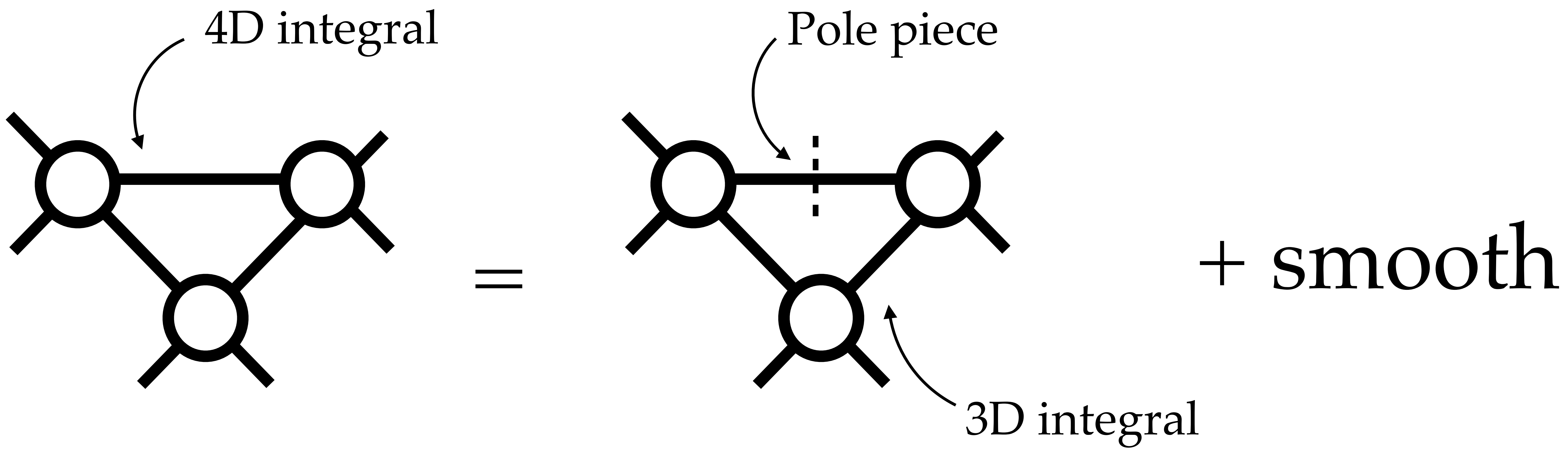
Three-hadron systems

- Sum over all **3 ➔ 3** amputated diagrams

$i\mathcal{M}_3 =$  $=$  $+$  $+ \cdots$

Reducing 4D to 3D

- Reducing from 4D to 3D while preserving singularities
- Remember, physical singularities are due to on-shell intermediate particles
- Let's consider a useful example:



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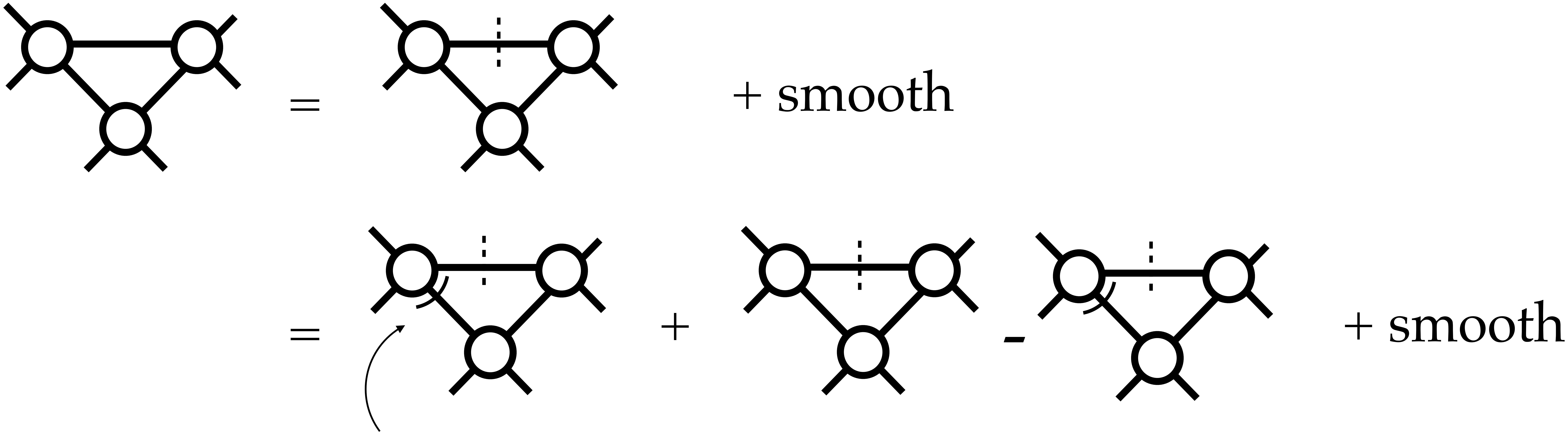

place the vertex on shell
but not the pole…i.e. it's still a 3D integral.

Reducing 4D to 3D

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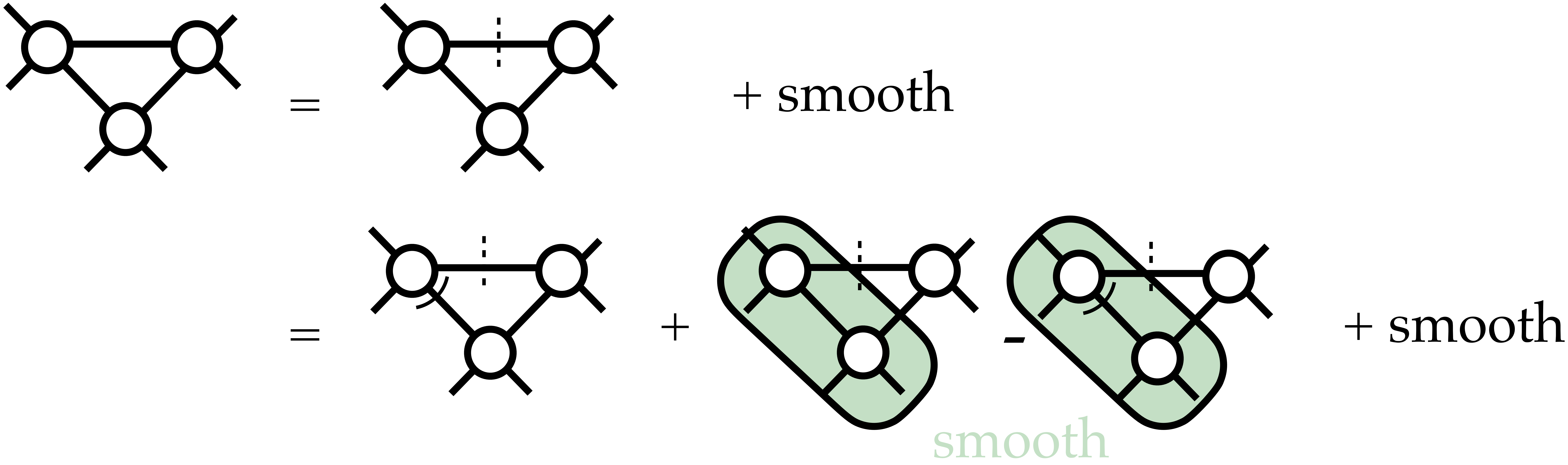

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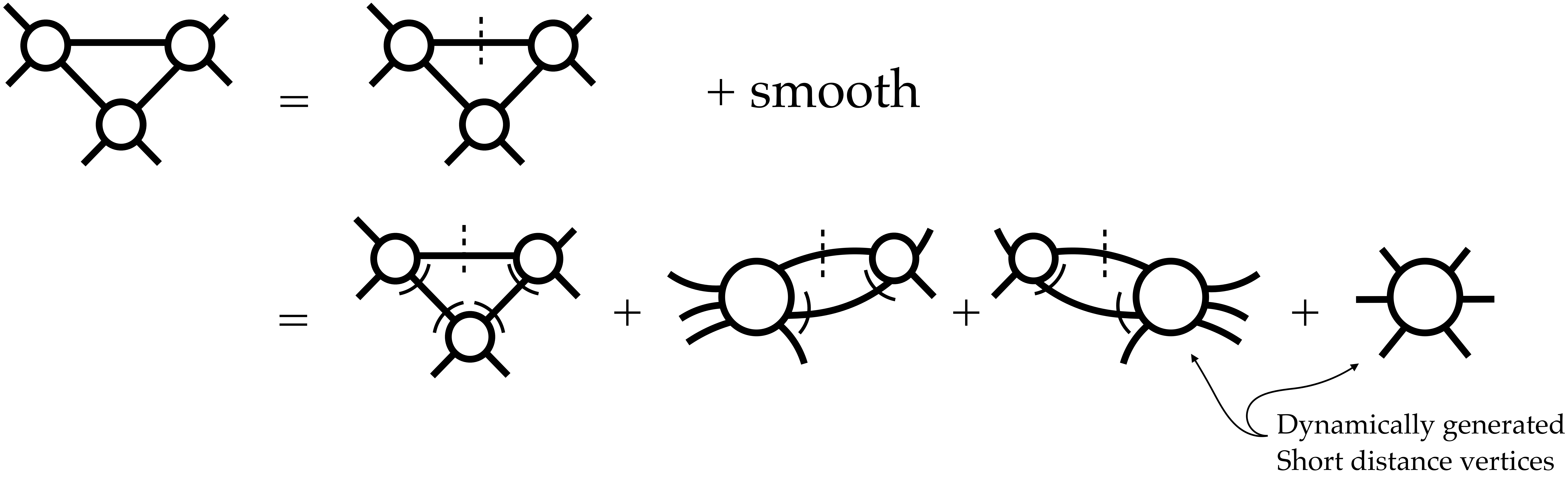
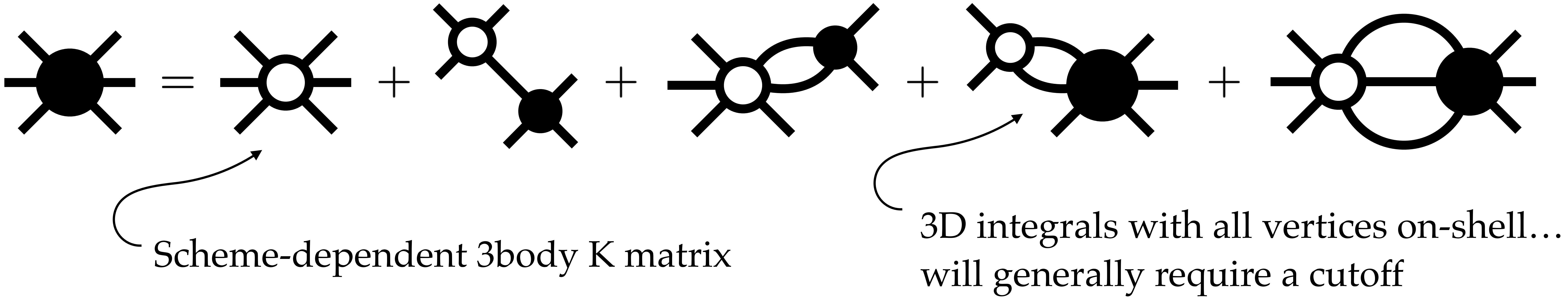


Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

$$i\mathcal{M}_3 =$$



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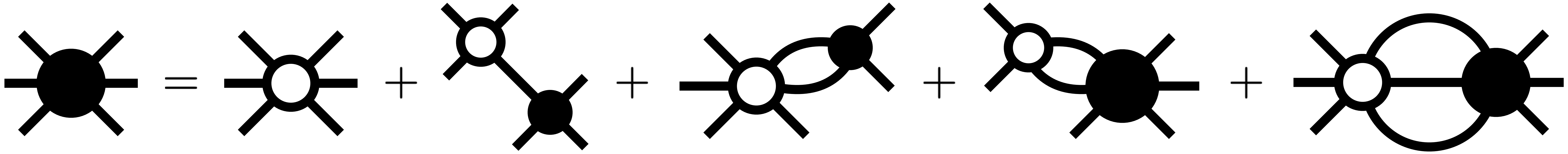
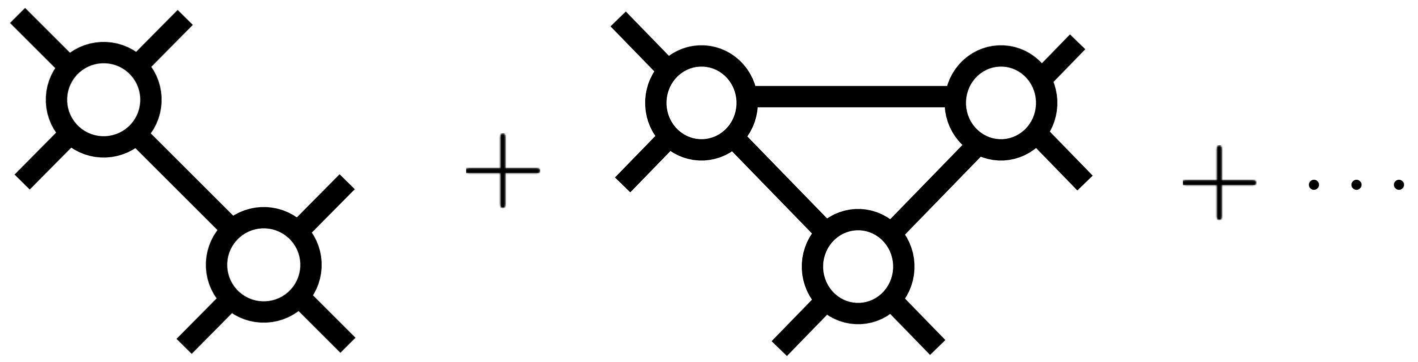


$$= i\mathcal{D} + i\mathcal{M}_{3,\mathrm{df}}$$

Sum over all ladder diagrams



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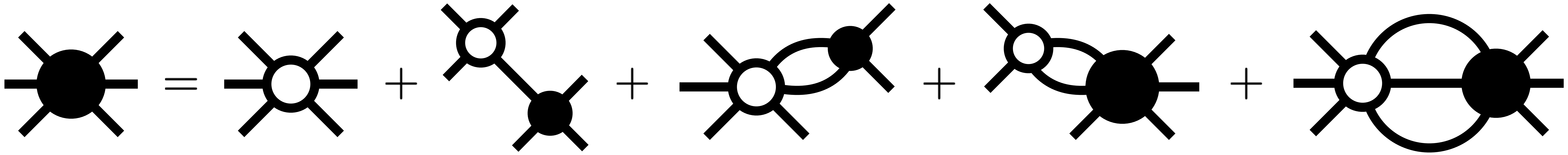
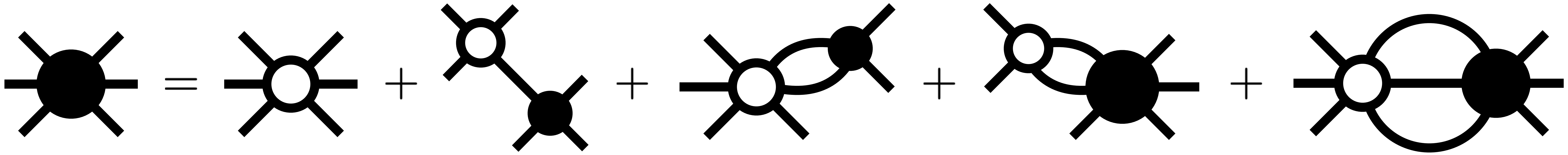


$$= i\mathcal{D} + i\mathcal{M}_{3,\mathrm{df}}$$

The rest [df = “divergence free”]

Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

$i\mathcal{M}_3 =$ 

$$= i\mathcal{D} + i\mathcal{M}_{3,\text{df}}$$

- Satisfies integral equation

$$\mathcal{D} = -\mathcal{M}_2\, G\, \mathcal{M}_2 - \int \mathcal{M}_2\, G\, \mathcal{D}$$

Jackura et al (2020)
M. Dawid et al (2023)

Three-hadron systems

- After some more work, one can show that the full amplitude satisfies a 3D integral equation

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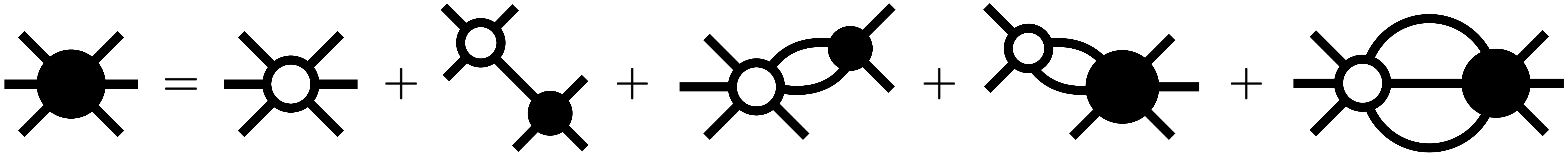

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Three-hadron systems

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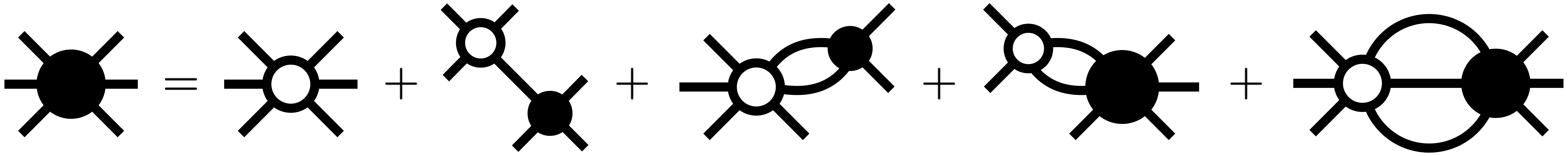


$$= i\mathcal{D} + i\mathcal{M}_{3,\mathrm{df}}$$

- Satisfies integral equation

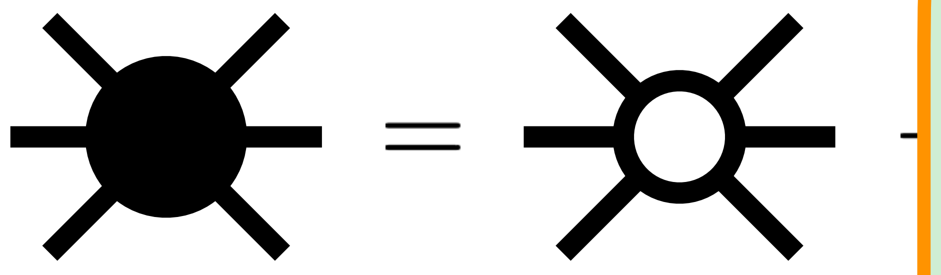
$$\mathcal{D} = -\mathcal{M}_2\, G\, \mathcal{M}_2 - \int \mathcal{M}_2\, G\, \mathcal{D}$$

$$\mathcal{M}_{3,\mathrm{df}}(\mathbf{p},\mathbf{k}) = \int_{p'}\int_{k'} \mathcal{L}(\mathbf{p},\mathbf{p'}) \cdot \mathcal{T}(\mathbf{p'},\mathbf{k'}) \cdot \mathcal{L}(\mathbf{k'},\mathbf{k}) \qquad \mathcal{L} = \frac{1}{3} + \mathcal{M}_2\rho - \mathcal{D}\rho$$

Three-hadron systems

- Sum over all 3 → 3 amputated diagrams

$i\mathcal{M}_3 =$



$= i\mathcal{D} + i\mathcal{M}_{3,\mathrm{df}}$

- Satisfies integral equation

- **Need to numerical solve these equations**
  - **Note: $\mathcal{D}$ and $\mathcal{T}$ are 3D integrals equations**
  - **Need to project to to definite angular momentum and parity**
  - **Integration kernel generally singular**
    - **Singular behavior mainly driven by the OPE propagator, $\mathcal{G}$**

$$\mathcal{D} = -\mathcal{M}_2\, G\, \mathcal{M}_2 - \int \mathcal{M}_2\, G\, \mathcal{D}$$

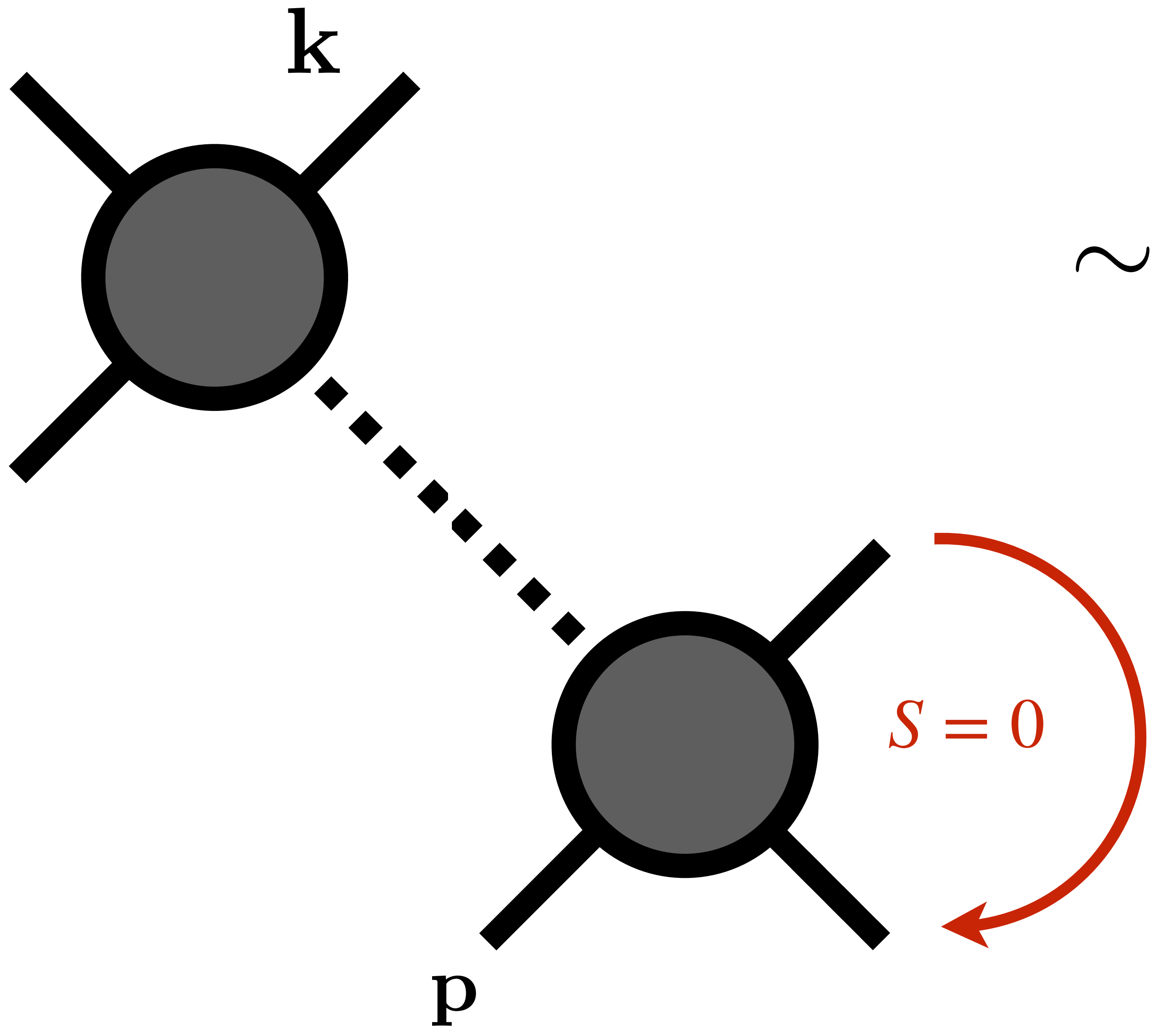
$$\mathcal{M}_{3,\mathrm{df}}(\mathbf{p},\mathbf{k}) = \int_{p'}\int_{k'} \mathcal{L}(\mathbf{p},\mathbf{p'})\cdot\mathcal{T}(\mathbf{p'},\mathbf{k'})\cdot\mathcal{L}(\mathbf{k'},\mathbf{k})$$

$$\mathcal{L} = \frac{1}{3} + \mathcal{M}_2\rho - \mathcal{D}\rho$$

$$\mathcal{T}(\mathbf{p},\mathbf{k}) = \mathcal{K}_3(\mathbf{p},\mathbf{k}) = \int_{p'}\int_{k'} \mathcal{K}_3(\mathbf{p},\mathbf{p'})\cdot\frac{\rho(p')}{2\omega_{p'}}\mathcal{L}(\mathbf{p'},\mathbf{k'})\cdot\mathcal{T}(\mathbf{k'},\mathbf{k})$$

OPE Partial wave projection

- Assuming s-wave two-body scattering



$$\sim G(\boldsymbol{p}, \boldsymbol{k}) = \frac{1}{(E - \omega_k - \omega_p) - (\boldsymbol{p} + \boldsymbol{k})^2 - m^2 + i\epsilon}$$

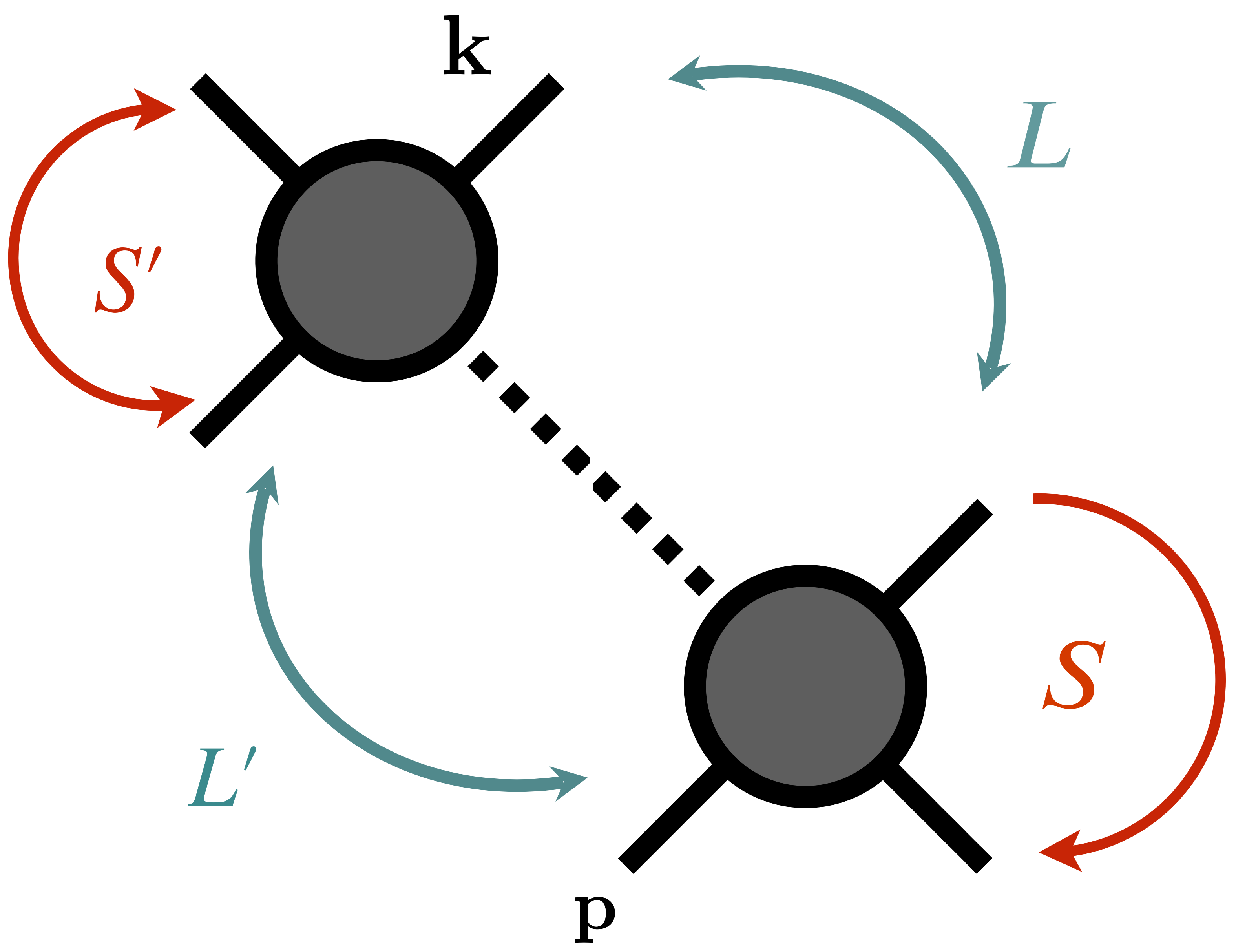
$$= \frac{1}{(E - \omega_k - \omega_p) - k^2 - p^2 - m^2 - 2pk\cos\theta + i\epsilon}$$

- After partial wave projecting to total J = 0:

$$\sim G(p, k) = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, G(\boldsymbol{p}, \boldsymbol{k}) = -\frac{1}{4pk}\log\frac{z_{pk} - 1}{z_{pk} + 1}$$

Non-zero angular momentum

- In general…assuming spinless particles



$$\mathcal{G}^{JP}_{L'S',LS}(p,k) = \mathcal{K}^{JP}_{\mathcal{G};L'S',LS}(p,k) + \mathcal{C}^{JP}_{L'S',LS}(p,k)\, Q_0(\zeta_{pk})$$

known kinematic functions

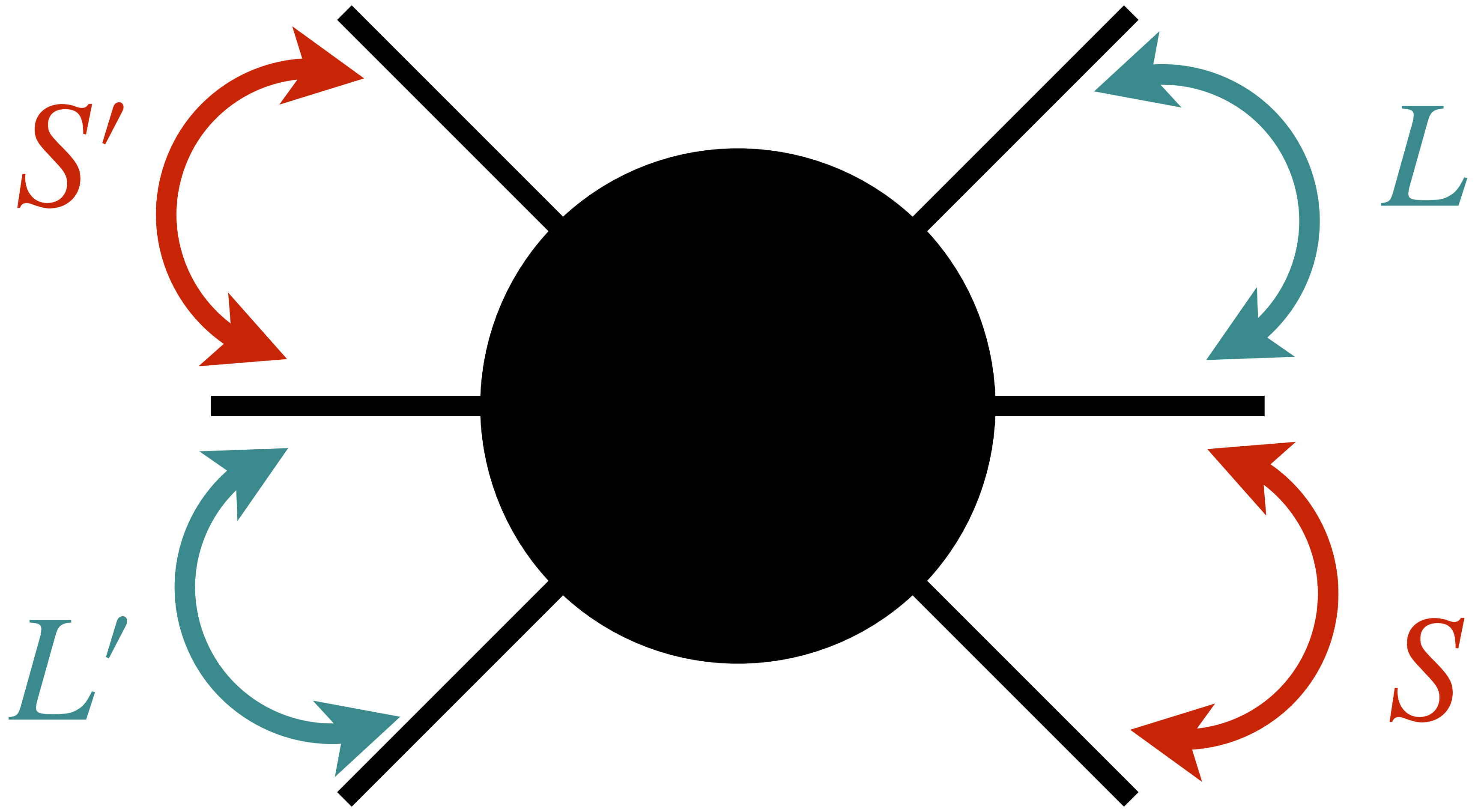
non-analyticity entirely encoded in

- $\mathcal{K}^{JP}_{\mathcal{G};L'S',LS}$ and $\mathcal{C}^{JP}_{L'S',LS}(p,k)$ are know kinematic functions that need to be generated for each channel

Jackura, Briceno (2023)

Partial wave projections

- In general:



$$= i \left[\mathcal{M}_3^{J^P}\right]_{L'S',LS}$$

Costa, Jackura, RB (2024)

Partial wave projections

- Partial wave projected amplitude

$$\mathcal{M}_3^{J^P}(p,k) = \mathcal{D}^{J^P}(p,k) + \mathcal{M}_{3,\mathrm{df}}^{J^P}(p,k)$$

$$\mathcal{D}^{J^P}(p,k) = \mathcal{D}_0^{J^P}(p,k) - \mathcal{M}_2(\sigma_p) \cdot \int_{k'} \mathcal{G}^{J^P}(p,k') \cdot \mathcal{D}^{J^P}(k',k)$$

- Two classes of K matrix are possible

1. Symmetric K matrix → $\mathcal{M}_{3,\mathrm{df}}^{J^P}(p,k) = \int_{p'} \int_{k'} \mathcal{L}^{J^P}(p,p') \cdot \mathcal{T}^{J^P}(p',k') \cdot \mathcal{R}^{J^P}(k',k)$
2. Asymmetric K matrix → $\widehat{\mathcal{M}}_{3,\mathrm{df}}^{J^P}(p,k) = \int_{p'} \int_{k'} \widehat{\mathcal{L}}^{J^P}(p,p') \cdot \widehat{\mathcal{T}}^{J^P}(p',k') \cdot \widehat{\mathcal{R}}^{J^P}(k',k)$

Costa, Jackura, RB (2024)

Partial wave projections

- Two classes of K matrix are possible

2. Asymmetric K matrix → $$\widehat{\mathcal{M}}^{J^P}_{3,\mathrm{df}}(p,k) = \int_{p'}\int_{k'} \widehat{\mathcal{L}}^{J^P}(p,p')\cdot\widehat{\mathcal{T}}^{J^P}(p',k')\cdot\widehat{\mathcal{R}}^{J^P}(k',k)$$

$$\left[\widehat{\mathcal{L}}^{J^P}(p,k)\right]_{L'S',LS} = \left[1-\mathcal{M}_{2,S'}(\sigma_p)\,\widetilde{\rho}(\sigma_p)\right]\delta_{L'L}\delta_{S'S}\,\frac{(2\pi)^2\omega_k}{k^2}\,\delta(p-k) - \mathcal{M}_{2,S'}(\sigma_p)\mathcal{G}^{J^P}_{L'S',LS}(p,k)$$
$$- \mathcal{D}^{J^P}_{L'S',LS}(p,k)\,\widetilde{\rho}(\sigma_k) - \sum_{L'',S''}\int_{k'} \mathcal{D}^{J^P}_{L'S',L''S''}(p,k')\,\mathcal{G}^{J^P}_{L''S'',LS}(k',k)$$

$$\left[\widehat{\mathcal{R}}^{J^P}(p,k)\right]_{L'S',LS} = \left[1-\widetilde{\rho}(\sigma_k)\,\mathcal{M}_{2,S'}(\sigma_k)\right]\delta_{L'L}\delta_{S'S}\,\frac{(2\pi)^2\omega_p}{p^2}\,\delta(p-k) - \mathcal{G}^{J^P}_{L'S',LS}(p,k)\mathcal{M}_{2,S}(\sigma_k)$$
$$- \widetilde{\rho}(\sigma_p)\,\mathcal{D}^{J^P}_{L'S',LS}(p,k) - \sum_{L'',S''}\int_{p'} \mathcal{G}^{J^P}_{L'S',L''S''}(p,p')\,\mathcal{D}^{J^P}_{L''S'',LS}(p',k)$$

$$\widehat{\mathcal{T}}^{J^P}(p,k) = \widehat{\mathcal{K}}^{J^P}_3(p,k) - \int_{p'}\int_{k'} \widehat{\mathcal{K}}^{J^P}_3(p,p')\cdot\widehat{\mathcal{F}}^{J^P}(p',k')\cdot\widehat{\mathcal{T}}^{J^P}(k',k)$$

$$\widehat{\mathcal{F}}^{J^P}(p,k) \equiv \widetilde{\rho}(\sigma_p)\,\widehat{\mathcal{L}}^{J^P}(p,k) + \int_{k'}\mathcal{G}(p,k')\cdot\widehat{\mathcal{L}}^{J^P}(k',k)$$

Costa, Jackura, RB (2024)

Separable K matrix

- We consider a parametrization for which the kinematic dependence factorizes:

$$\left[\widehat{\mathcal{K}}_3^{J^P}(p,k)\right]_{L'S',LS} = [h(p)]_{L'S'} \left[\widetilde{\mathcal{K}}_3^{J^P}(s)\right]_{L'S',LS} [h(k)]_{LS}$$

Separable K matrix

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leads to an also factorizable $\widehat{\mathcal{T}}^{J^P}$

$$\widetilde{\mathcal{T}}^{J^P}(s) = \frac{1}{1+\widetilde{\mathcal{K}}_3^{J^P}(s)\cdot\widetilde{\mathcal{F}}^{J^P}(s)}\cdot\widetilde{\mathcal{K}}_3^{J^P}(s)$$

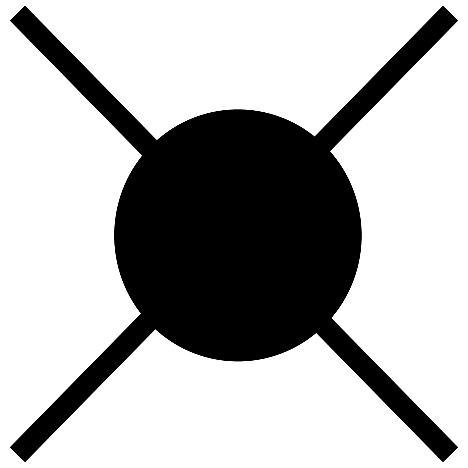
$$\widetilde{\mathcal{F}}^{J^P}(s) = \int_p\int_k \widetilde{\mathcal{R}}^{J^P}(s,p)\cdot\Gamma^{J^P}(p,k)\cdot h(k)$$

$$\Gamma^{J^P}(p,k) = \frac{(2\pi)^2\,\omega_k}{k^2}\,\delta(p-k)\,\widetilde{\rho}(\sigma_p) + \mathcal{G}^{J^P}(p,k)$$

From 3body → 2body

- $\mathcal{M}_3^{J^P}$ is effectively described as a 2-body system (pair+spectator)
  - pair can become bound: $b_k + \varphi_k \to b_p + \varphi_p$

$\mathcal{M}_2(\sigma_k)$ =

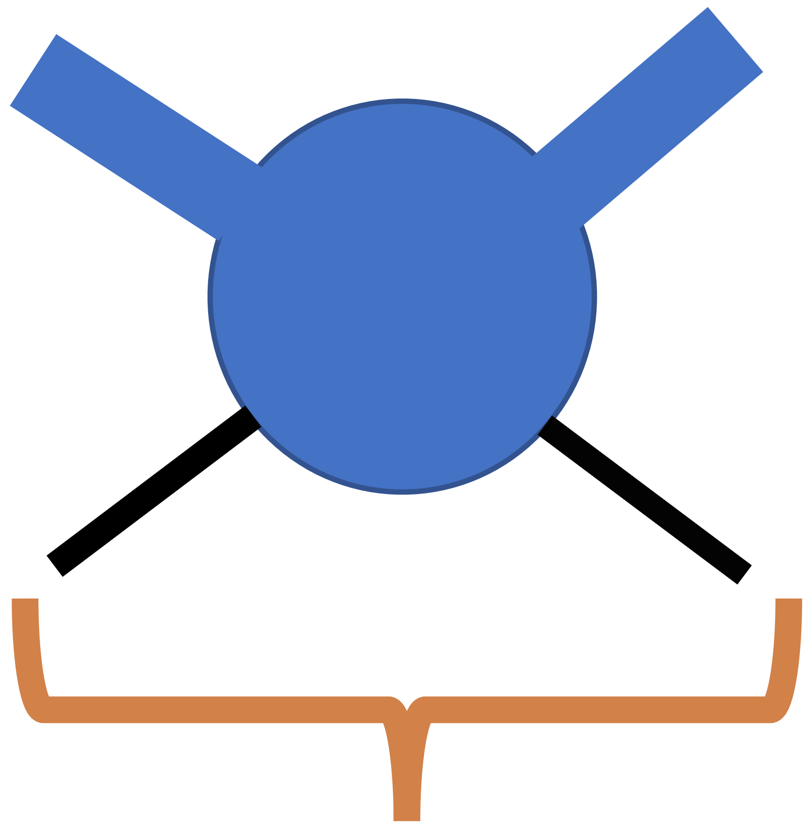


$\sim$

$$\sim \quad -\frac{g_{k,b}^2}{\sigma_k - \sigma_{k,b}}$$

- Bound-state spectator amplitude:

$$\mathcal{M}_3^{J^P} \sim \left(-\frac{g_{p,b}}{\sigma_p - \sigma_{p,b}}\right)$$



$$\left(-\frac{g_{k,b}}{\sigma_k - \sigma_{k,b}}\right)$$

$$\mathcal{M}_{\varphi b}^{J^P} = \lim_{\substack{\sigma_p \to \sigma_{p,b} \\ \sigma_k \to \sigma_{k,b}}} \frac{(\sigma_p - \sigma_{p,b})(\sigma_k - \sigma_{k,b})}{g_{p,b}\, g_{k,b}} \, \mathcal{M}_3^{J^P}$$

Toy models for 3π systems

$S \leq 1$ and $L \leq 2$

$$[\pi\pi]^I_\ell + \ \pi \ \to \ [\pi\pi]^{I'}_{\ell'} + \pi$$

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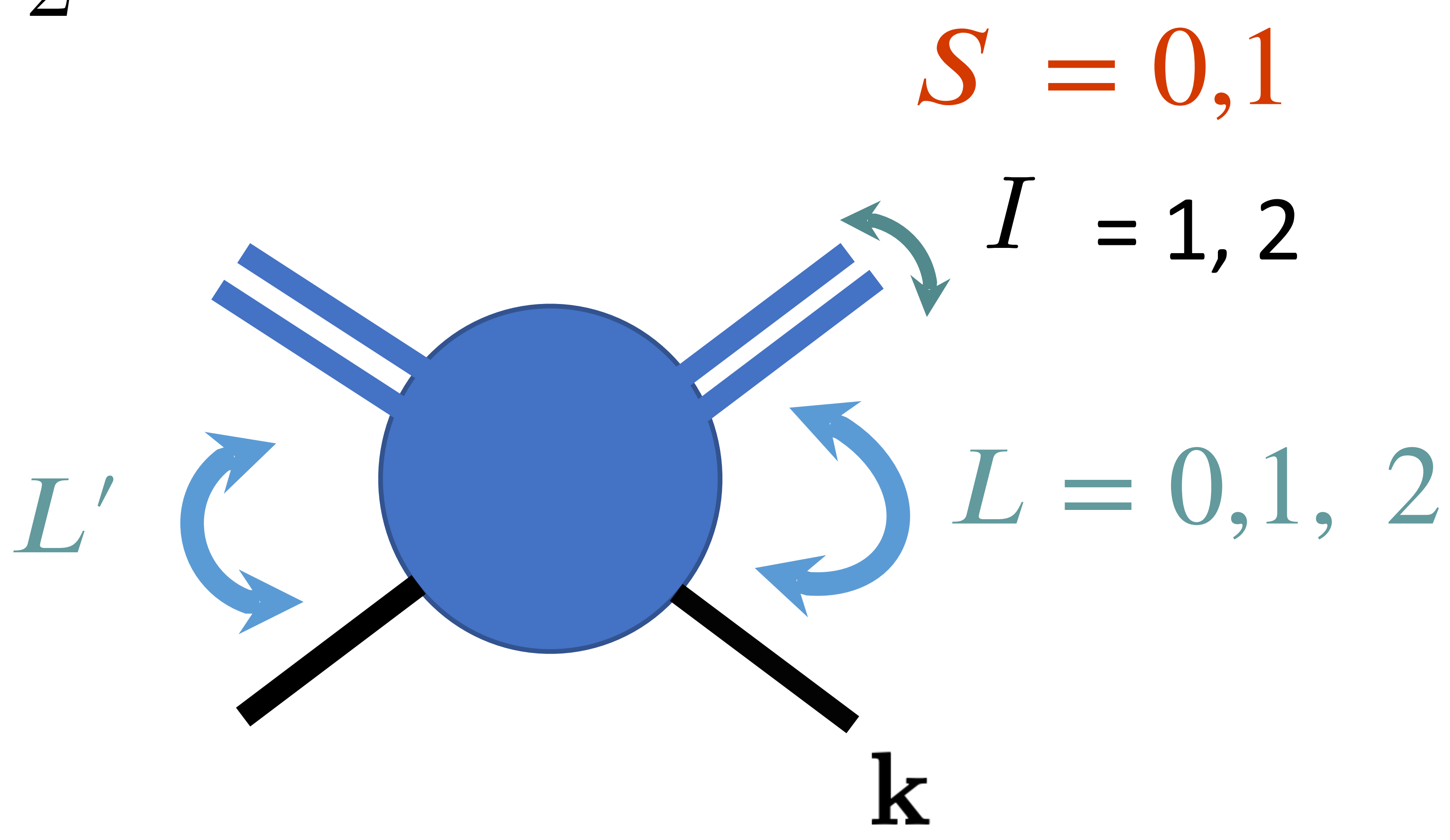
$T(J^P) = 2(1^+)$ ***channel with stable*** ρ



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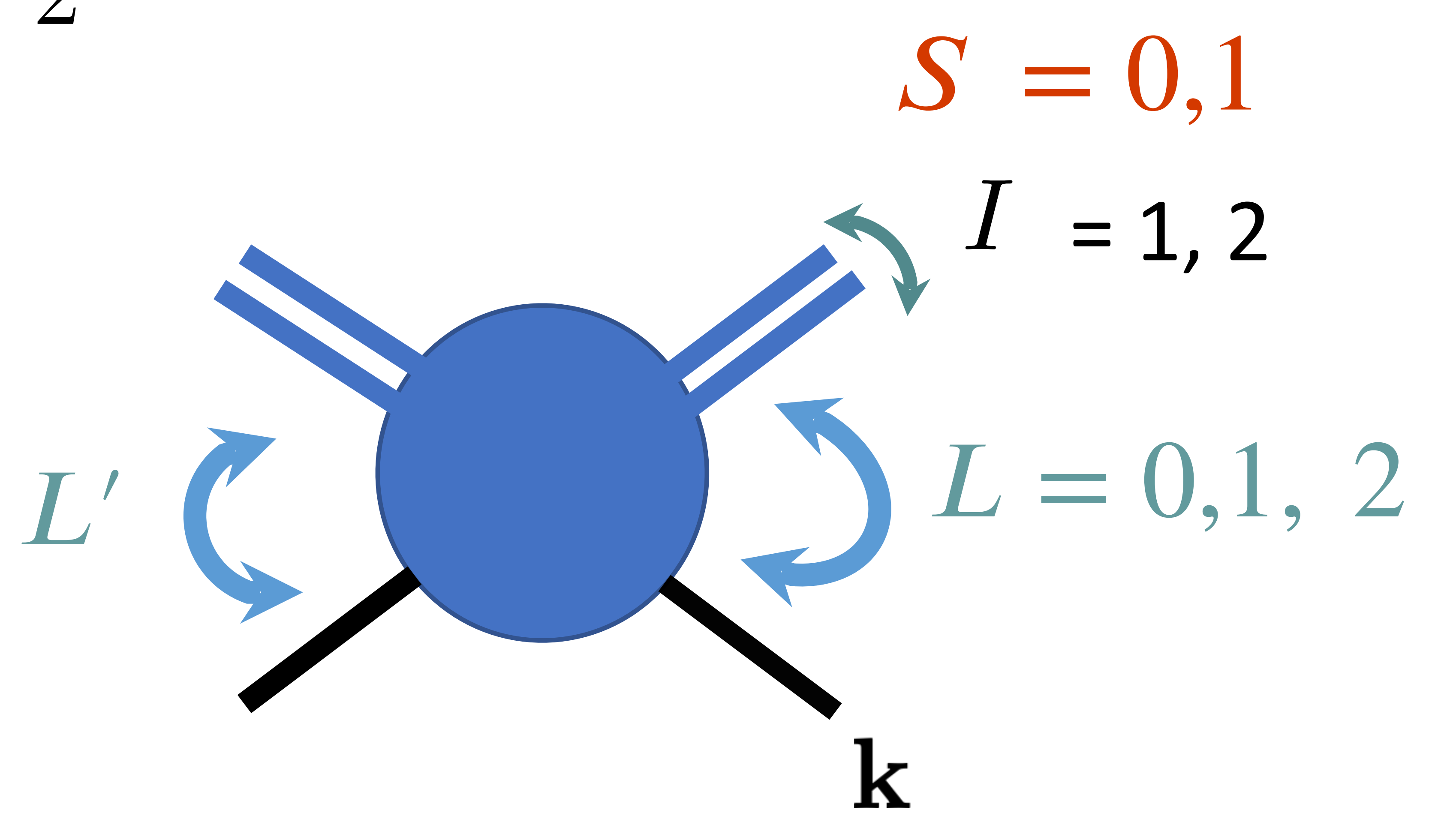
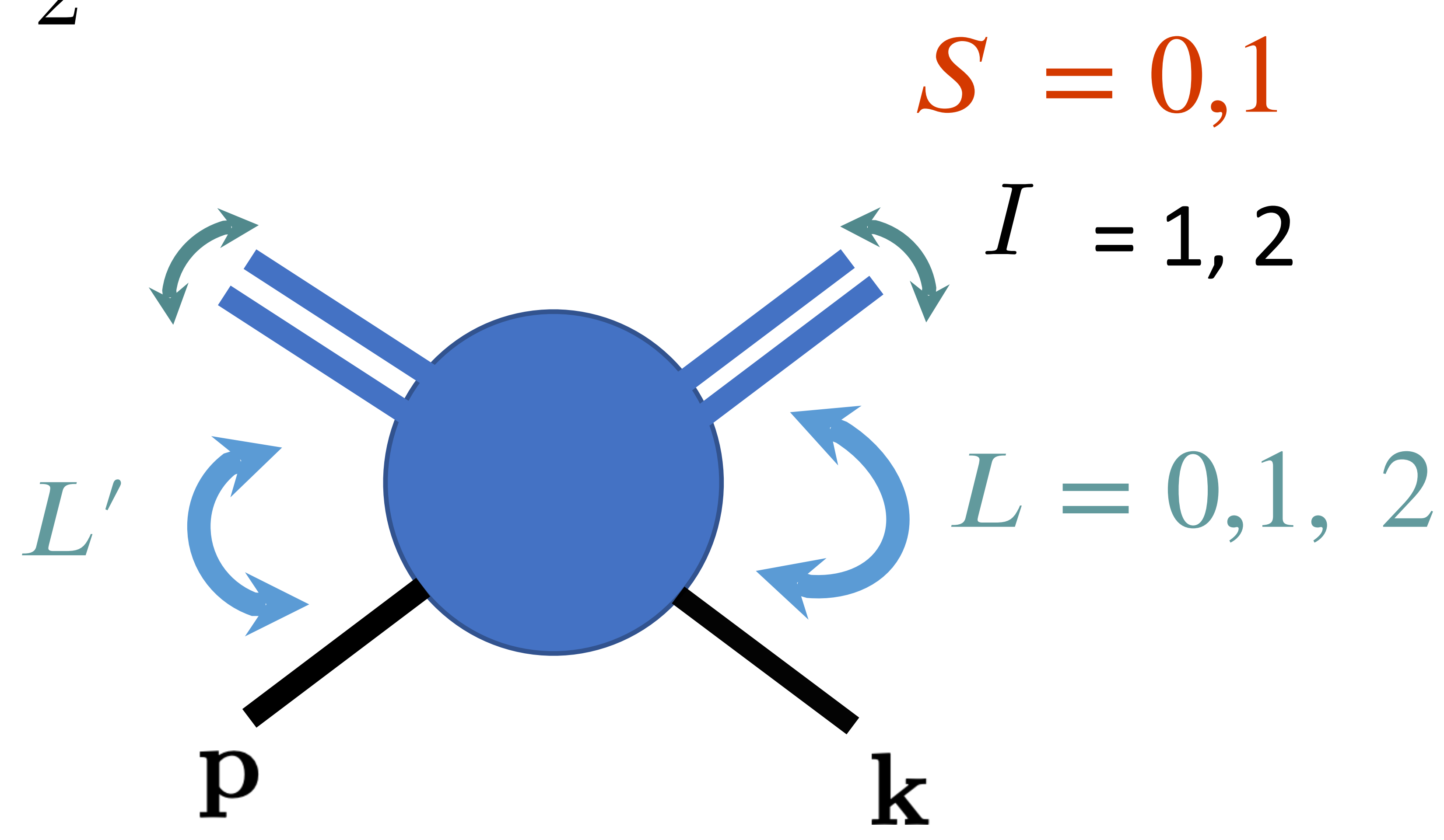


$$([\pi\pi]^2_S\pi)_P,\ ([\pi\pi]^1_P\pi)_S,\ ([\pi\pi]^1_P\pi)_D$$

Toy models for 3π systems

$S \leq 1$ and $L \leq 2$



$$[\pi\pi]^I_\ell + \pi \to [\pi\pi]^{I'}_{\ell'} + \pi$$

$T(J^P) = 2(1^+)$ ***channel with stable*** ρ

$$([\pi\pi]^2_S\pi)_P, \; ([\pi\pi]^1_P\pi)_S, \; ([\pi\pi]^1_P\pi)_D$$

Can couple to the rho

Bound state model

- Can get a two-body bound state for S- and P-waves by parametrizing $\mathcal{M}_2$ via the phase shift:

$$\mathcal{M}_{2,IS}(\sigma_k) = \frac{16\pi\sqrt{\sigma_k}}{q_k^\star \cot\delta_{S,I} - i q_k^\star} \qquad \left(q_k^* = \sqrt{\sigma_k/4 - m^2}\right)$$

- For a S-wave 2 body amplitude, we can use a LO ERE:

$$q_k^\star \cot\delta_{0,I} = -\frac{1}{a_{0,I}}$$

  - For a P-wave bound state, LO ERE leads to unphysical poles

- Instead, for P-waves, we use

$$q_k^\star \cot\delta_{1,1} = \frac{(m_{\mathrm{BW}}^2 - \sigma_k)}{\sqrt{\sigma_k}\,\Gamma_1^{\mathrm{BW}}(\sigma_k)} \qquad \Gamma_1^{\mathrm{BW}}(\sigma_k) = \frac{g_{\mathrm{BW}}^2}{6\pi\sigma_k} q_k^{\star 2}$$

Solving integral equations

- Deform contour to miss singularities and discretize momenta
  - sometimes useful / / sometimes critical

- Discretize momenta: $$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$

$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\max}} K(p', s, q)\, d(q, s, p)$$

[contains pole, logarithmic and **square root cuts]**

Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)

Solving integral equations

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- Use linear algebra:

$$[1 + \boldsymbol{K}] \cdot \vec{d}_{\mathbf{sol}}(s, p) = -\vec{G}(s, p)$$

Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)

Solving integral equations

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$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\rm max}} K(p', s, q)\, d(q, s, p)$$

- Use linear algebra:

$$[1 + \boldsymbol{K}] \cdot \vec{d}_{\bf sol}(s, p) = -\vec{G}(s, p)$$

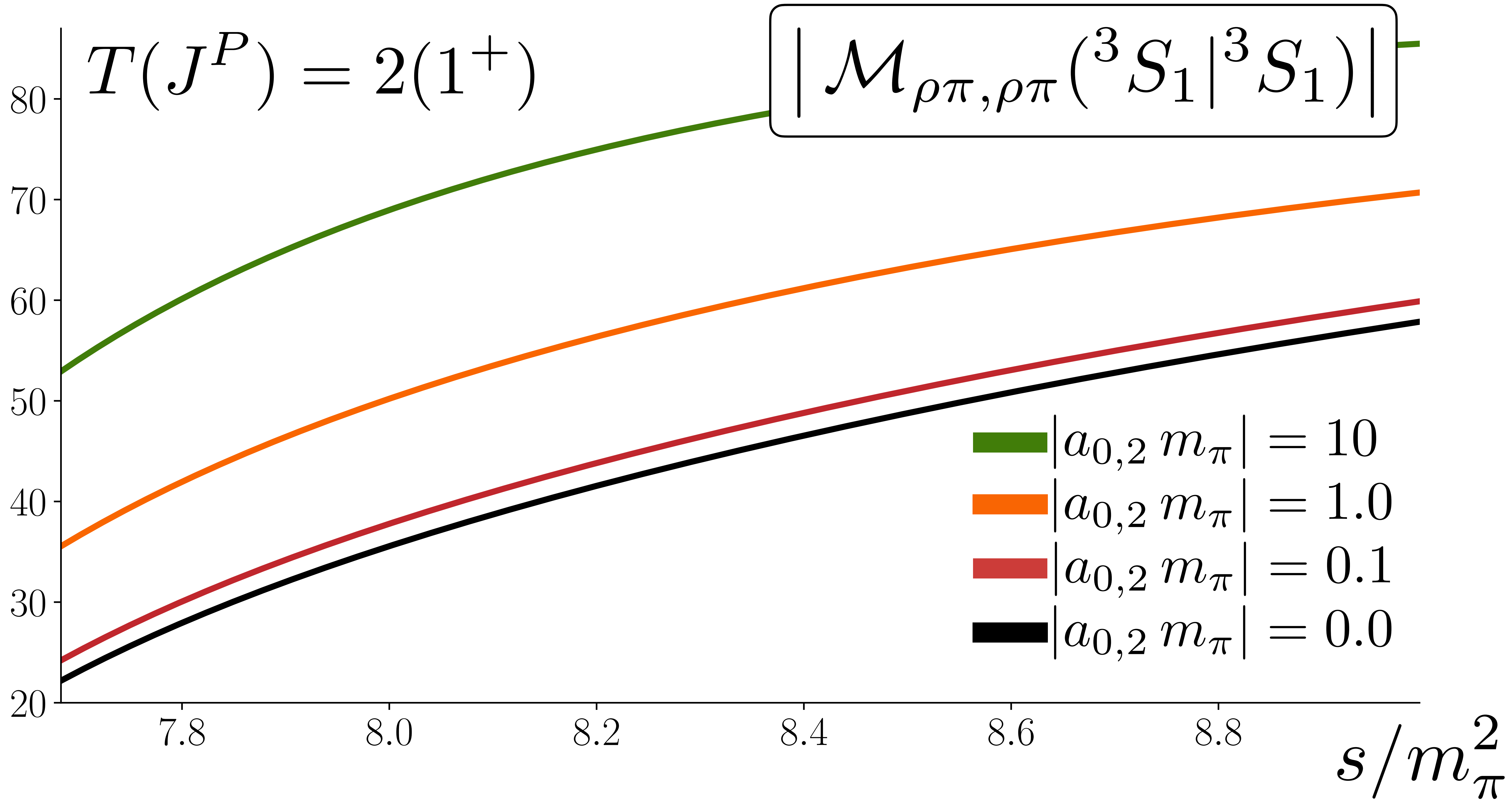
- Use integral equation to interpolate or extrapolate:

$$d(p', s, p) \approx -G(p', s, p) - \vec{K}(p', s) \cdot \vec{d}_{\rm sol}(s, p)$$

Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)

$T(J^P) = 2(1^+)$ ***channel with stable*** ρ

$$d^{2(1^+)} = \begin{pmatrix} d_{\rho\pi,\rho\pi}(^3S_1|^3S_1) & d_{\rho\pi,\rho\pi}(^3S_1|^3D_1) & d_{\rho\pi,t\pi}(^3S_1|^1P_1) \\ & d_{\rho\pi,\rho\pi}(^3D_1|^3D_1) & d_{\rho\pi,t\pi}(^3D_1|^1P_1) \\ & & d_{t\pi,t\pi}(^1P_1|^1P_1) \end{pmatrix}$$



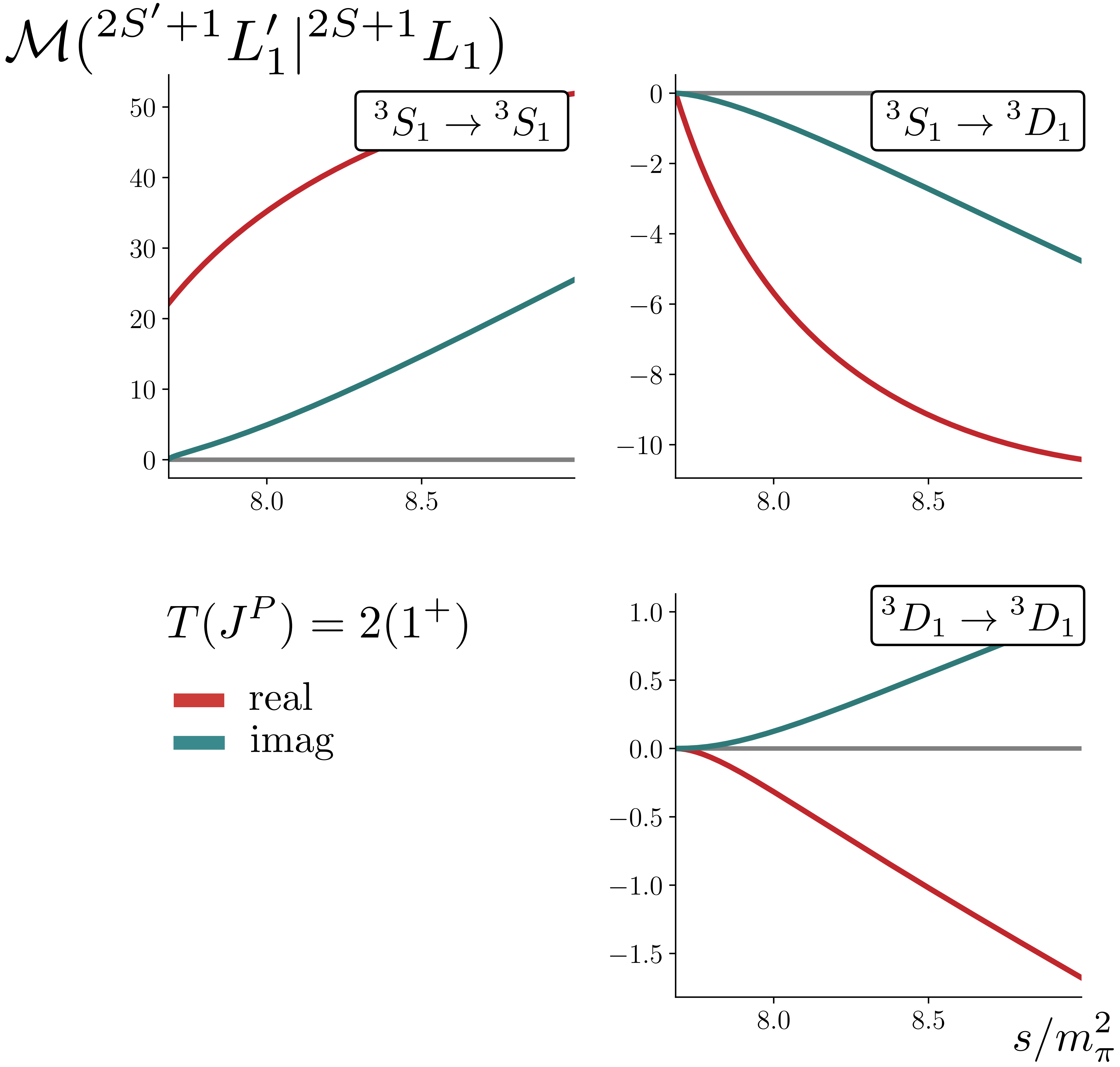
$T(J^P) = 2(1^+)$ *channel with stable* ρ

- Setting $a_{0,2} = 0$

$$\mathcal{M}_{\varphi b}^{2(1^+)} = \begin{pmatrix} \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3S_1) & \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3D_1) \\ & \mathcal{M}_{\rho\pi,\rho\pi}({}^3D_1|{}^3D_1) \end{pmatrix}$$

- Satisfy 2-body unitarity condition
- Satisfy threshold behavior:

$$\mathcal{M}_{\rho\pi,\rho\pi}({}^3L'_1|{}^3L_1) \sim q_{\rho\pi}^{L'+L}$$



$T(J^P) = 1(1^+)$ ***channel with stable*** σ ***and*** ρ

$$([\pi\pi]_S^{0,2}\pi)_P,\ ([\pi\pi]_P^1\pi)_S,\ ([\pi\pi]_P^1\pi)_D$$

$T(J^P) = 1(1^+)$ ***channel with stable*** σ ***and*** ρ

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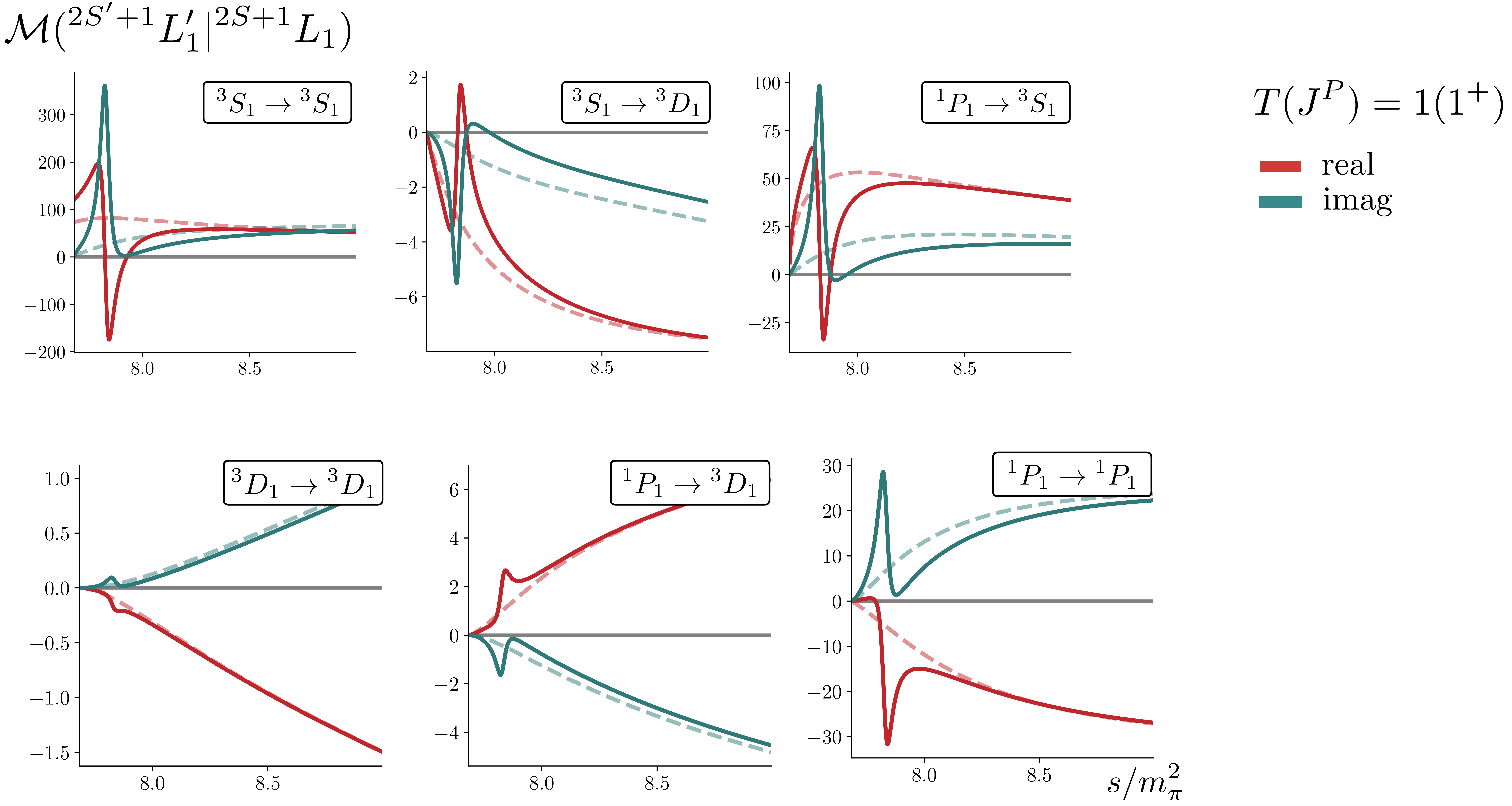
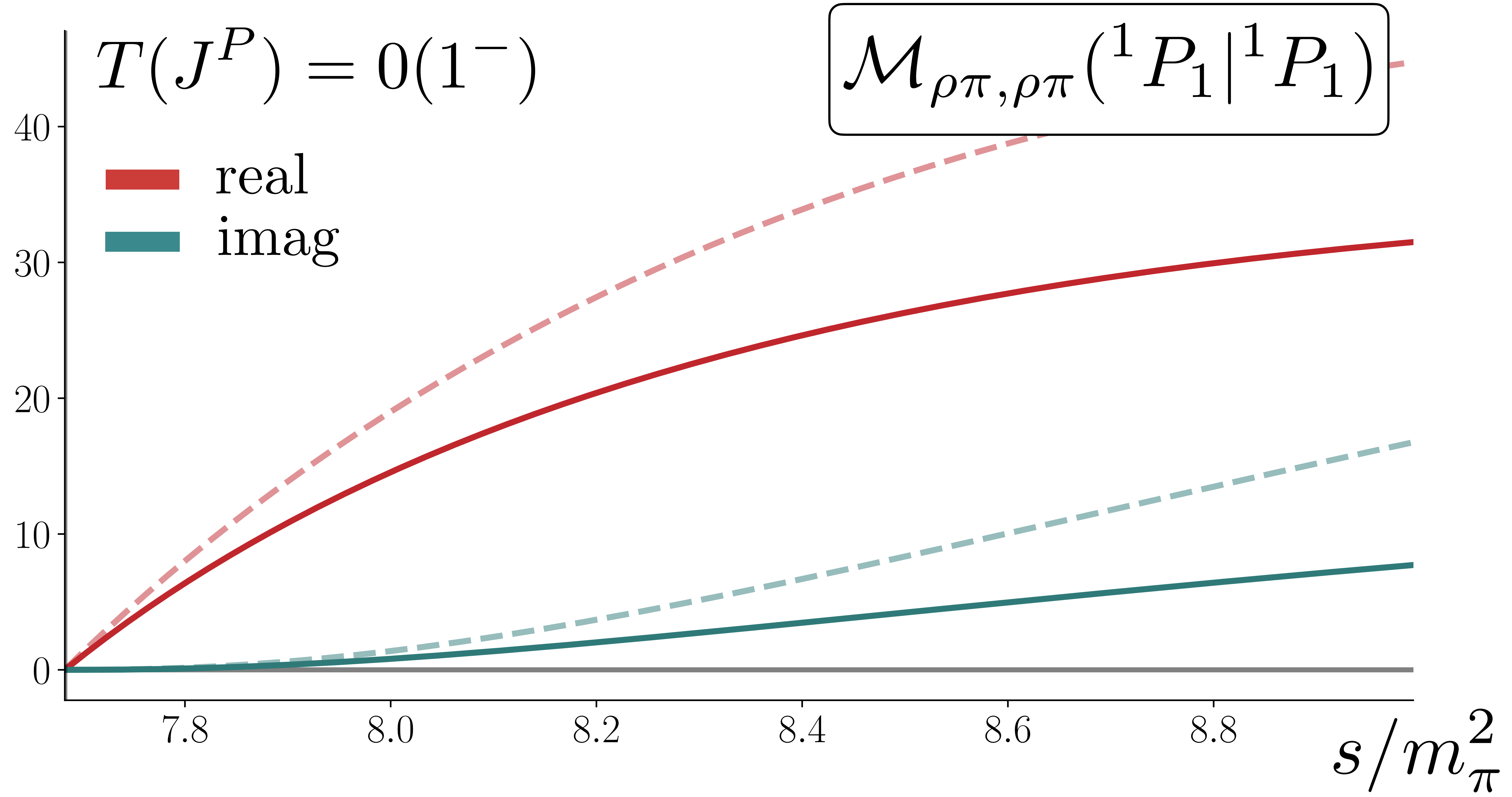
$$\mathcal{M}_{\varphi b}^{1(1^+)} = \begin{pmatrix} \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3S_1) & \mathcal{M}_{\rho\pi,\rho\pi}({}^3S_1|{}^3D_1) & \mathcal{M}_{\rho\pi,\sigma\pi}({}^3S_1|{}^1P_1) \\ & \mathcal{M}_{\rho\pi,\rho\pi}({}^3D_1|{}^3D_1) & \mathcal{M}_{\rho\pi,\sigma\pi}({}^3D_1|{}^1P_1) \\ & & \mathcal{M}_{\sigma\pi,\sigma\pi}({}^1P_1|{}^1P_1) \end{pmatrix}$$



$\mathcal{M}({}^{2S'+1}L'_1|{}^{2S+1}L_1)$
$T(J^P) = 1(1^+)$
real
imag
${}^3S_1 \to {}^3S_1$
${}^3S_1 \to {}^3D_1$
${}^1P_1 \to {}^3S_1$
${}^3D_1 \to {}^3D_1$
${}^1P_1 \to {}^3D_1$
${}^1P_1 \to {}^1P_1$
s/m_π^2

$T(J^P) = 0(1^-)$ ***channel with stable*** ρ



Summary & Perspectives

- [x] Overview on 2- and 3-body scattering amplitudes
- [x] Integral equations for PW projected 3-body amplitudes
- [x] Factorizable K-matrices
- [x] Toy model calculations for 3pi-systerms

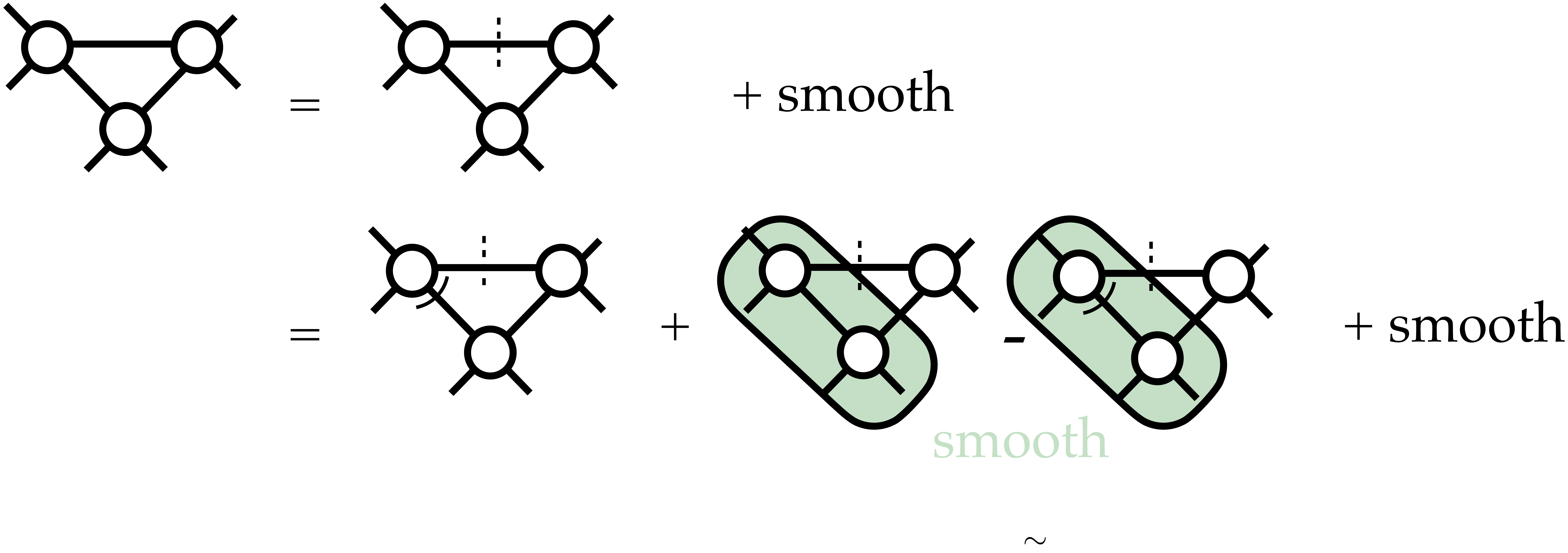
To do:

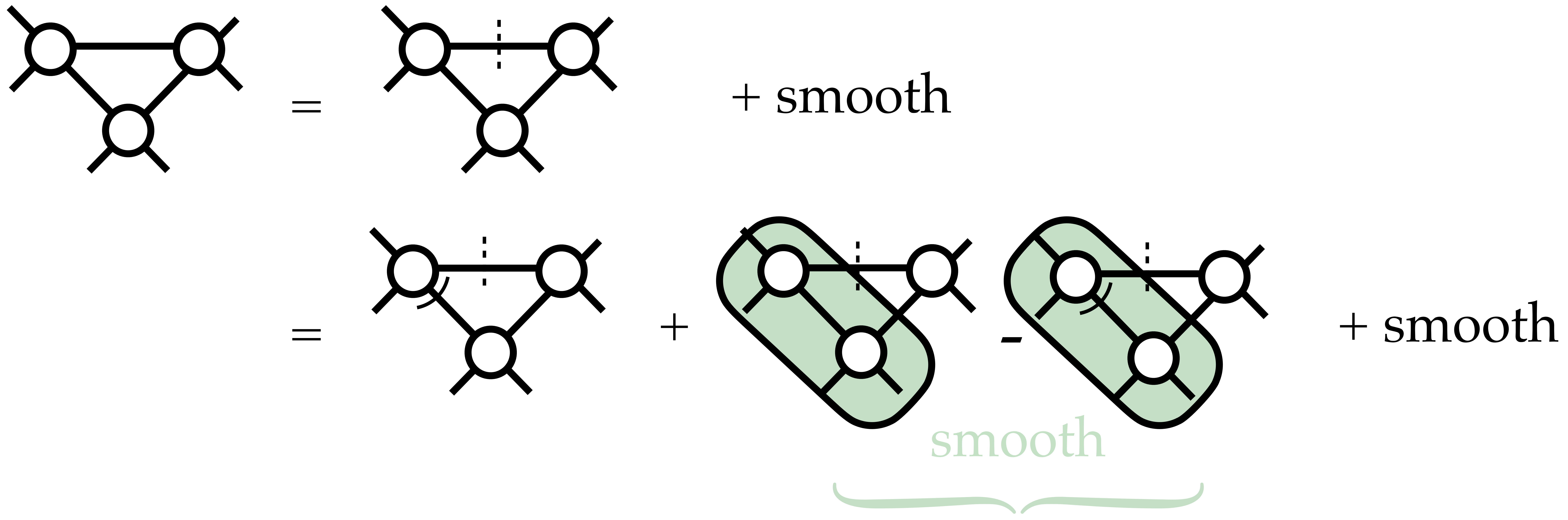
- [ ] Generalization to particles with spin, including 3N
- [ ] Checks on the finite-volume formalism (see Raúl's talk)

Backup

Reducing 4D to 3D

- Reducing from 4D to 3D while preserving singularities
- Remember, physical singularities are due to on-shell intermediate particles
- Let’s consider a useful example:

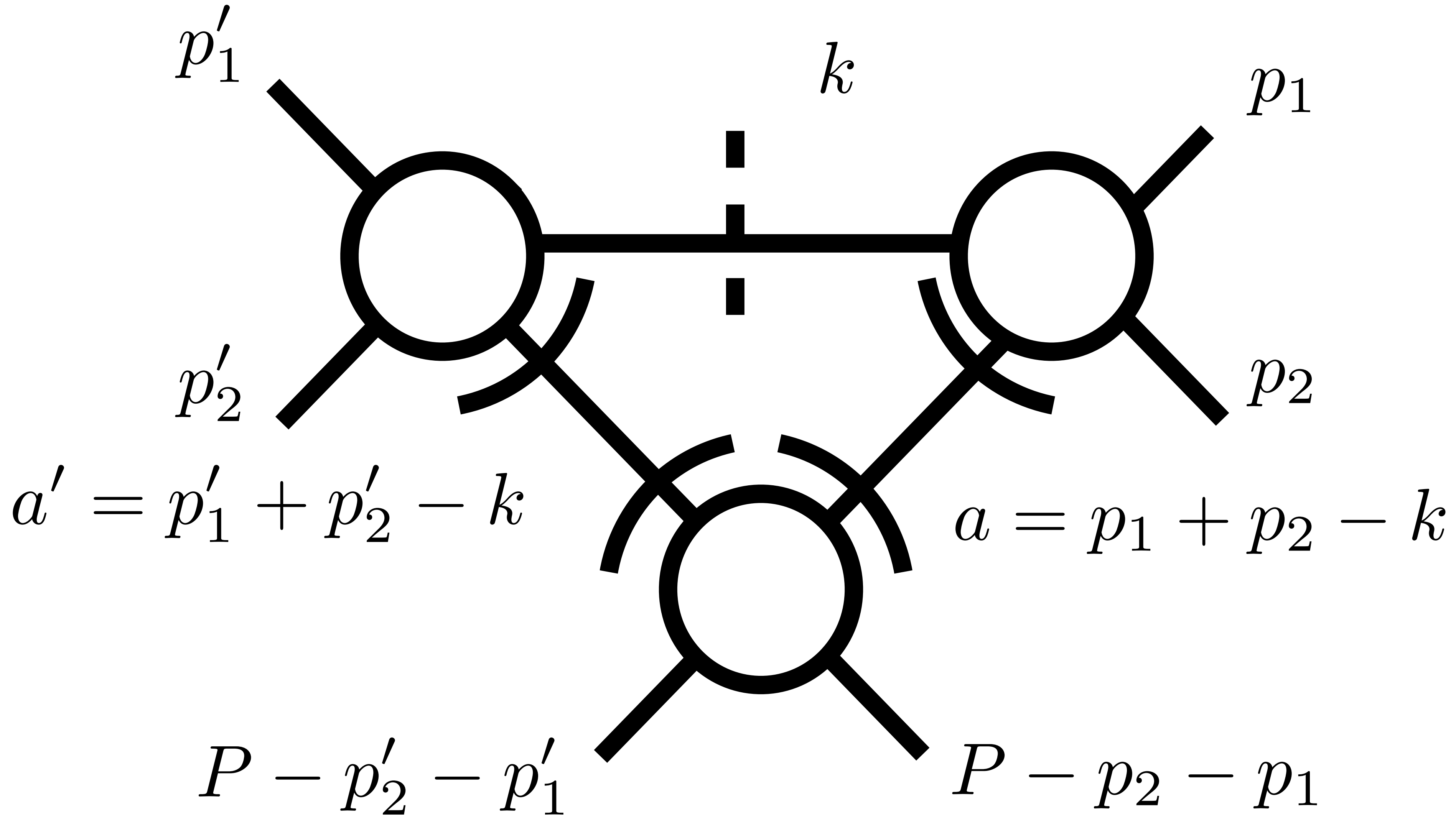




$$\sim \quad \frac{B(p^2) - B(m^2)}{p^2 - m^2}$$

$$\approx \frac{B(m^2) + B'(m^2)(p^2 - m^2) - B(m^2)}{p^2 - m^2}$$

$$\approx B'(m^2)$$



$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} iB_{\mathrm{on}}((p_1' + p_2')^2, \hat{k} \cdot \hat{p}_1') \frac{i}{a'^2 - m^2} iB_{\mathrm{on}}((P-k)^2, \hat{a} \cdot \hat{a}') \frac{i}{a^2 - m^2} iB_{\mathrm{on}}((p_1 + p_2)^2, \hat{k} \cdot \hat{p}_1)$$

OPE

$$\mathcal{G}_{\ell'\lambda',\ell\lambda}(\mathbf{p},\mathbf{k}) \equiv \frac{\mathcal{H}^{(\ell'\ell)}_{\lambda'\lambda}(\mathbf{p},\mathbf{k})}{u_{pk} - m_e^2 + i\epsilon} \; ;$$

$$\mathcal{H}^{(\ell'\ell)}_{\lambda'\lambda}(\mathbf{p},\mathbf{k}) \equiv \left(\frac{k_p^\star}{q_p^\star}\right)^{\ell'} 4\pi \, Y^*_{\ell'\lambda'}(\hat{\mathbf{k}}_p^\star) Y_{\ell\lambda}(\hat{\mathbf{p}}_k^\star) \left(\frac{p_k^\star}{q_k^\star}\right)^{\ell} \; .$$