Slightly revised title:

3-, 4-, and 5-particle physics near unitarity, the faux-Efimov effect, and beyond

Chris Greene, Purdue University

Main collaborators: (now postdoc) Michael Higgins and Yu-Hsin Chen (now faculty in Taiwan)

A recent group photo→

This work is primarily supported by NSF. Thanks!

Plan for this talk:

- The adiabatic hyperspherical toolkit, nuts and bolts
- Key results about N-body continua, building intuition, N=3,4,5…
- **Application #1:** nonexistence of 3-neutron and 4-neutron resonances (with Kievsky and Viviani, led by Michael Higgins)
- **Application #2:** Threshold laws near unitarity (Y-H Chen,MH)
- **Application #3:** how Efimov physics operates for general N-body continuum states, and the faux-Efimov effect (Y-H Chen,MH)
- **Application #4:** 5-body continuum physics with identical bosons and in nuclear systems (w/MH)
- Conclusions

Strategy of the adiabatic hyperspherical representation: convert the partial differential Schroedinger equation into an infinite set of coupled ordinary differential equations:

First, we set:

$$\psi = R^{(3(N-1)-1)/2}\Psi$$

(in order to eliminate 1st order- R-derivatives in K.E. operator)

To solve:

$$\left[-\frac{1}{2\mu}\frac{\partial^2}{\partial R^2}+\frac{\Lambda^2}{2\mu R^2}+\frac{(3N-6)(3N-4)}{8\mu R^2}+V(R,\theta,\varphi)\right]\psi_E=E\psi_E$$

Now solve the fixed-R Schroedinger equation, for eigenvalues $U_n(R)$:

$$\left[\frac{\Lambda^2}{2\mu R^2}+\frac{15}{8\mu R^2}+V(R,\theta,\varphi)\right]\Phi_\nu(R;\Omega)=U_\nu(R)\Phi_\nu(R;\Omega)$$

Next expand the desired solution into the complete set of eigenfunctions

$$\psi_E(R,\Omega)=\sum_\nu F_{\nu E}(R)\Phi_\nu(R;\Omega)$$

Look Ma, no Λ!

And the original T.I.S.Eqn. is transformed into the following set (coupled ODEs) which can be truncated on physical grounds, with the eigenvalues interpretable as adiabatic potential curves, in the Born-Oppenheimer sense.

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2}+U_\nu(R)\right]F_{\nu E}(R)-\frac{1}{2\mu}\sum_{\nu'}\left[2P_{\nu\nu'}(R)\frac{d}{dR}+Q_{\nu\nu'}(R)\right]F_{\nu' E}(R)=EF_{\nu E}(R)$$

The adiabatic hyperspherical representation then reduces the few- or many-particle problem to motion along potential curves in a single coordinate, with nonadiabatic couplings that mediate transitions between channels (potentials)

Note that there will be at least one potential curve converging at R→ infinity to each possible energy level of the fragmented system.

Note also that the noninteracting problem is exactly solvable and gives a separated Schroedinger equation with potential curves proportional to $1/R^2$ at all R values, which we like to write (for channel ν) as: which mimics our usual 3D form

$$U_\nu^{NI}(R) = \frac{\ell_\nu(\ell_\nu+1)}{2\mu R^2}$$

Eigenfunctions are the hyperspherical harmonics $Y_{\nu\mu}$(symmetrized appropriately)

Examples for different systems: 3 distinguishable particles or 3 identical bosons (0^+) has $\ell_\nu = \frac{3}{2}$

3 fermions, 2 spin up 1 spin down, $L^\pi=1^-$ symmetry, has $\ell_\nu = \frac{5}{2}$

3 identical spin-polarized fermions, $L^\pi=1^+$ symmetry, has $\ell_\nu = \frac{7}{2}$

And this controls the Wigner threshold law, i.e.: $S_{fi} \propto k^{\ell_\nu+1/2}$

Next suppose there are s-wave interactions among some subset of the particles, with scattering length *a*, represented by a Fermi pseudopotential ~ 1/R³:

$$V_{ij}^{\text{Fermi}} = \frac{2\pi a\hbar^2}{m}\delta(\vec{r_i} - \vec{r_j})$$

$$\left\langle \sum_{i<j} \frac{2\pi a\hbar^2}{m_{red}}\delta(\vec{r}_i - \vec{r}_j) \right\rangle = C\frac{a\hbar^2}{2\mu R^3}$$

Treating this in perturbation theory gives a modified adiabatic hyperradial potential curve, namely:

$$U_{\min}(R) \to \frac{\ell_{\text{NI}}(\ell_{\text{NI}}+1)\hbar^2}{2\mu R^2} + C\frac{a\hbar^2}{2\mu R^3}$$

Here the constant C depends on the symmetry L^π and the fermion/bose nature of particles and their number, e.g. for 3 or 4 fermions:

N	(L^π, S)	ν	l_{eff}	$\mathcal{C}_\nu$
3	$(1^-, 1/2)$	1	5/2	15.1(3)[a]
		2	9/2	15.2(3)[a]
4	$(0^+, 0)$	1	5	86.7(3)[a]
		2	7	156(3)[a]

M. D. Higgins, C. H. Greene, A. Kievsky, and M. Viviani, Comprehensive study of the three- and four-neutron systems at low energies, Phys. Rev. C **103**, 024004 (2021).

And for N identical bosons in the 0^+ symmetry, this long range potential can be computed analytically in general:

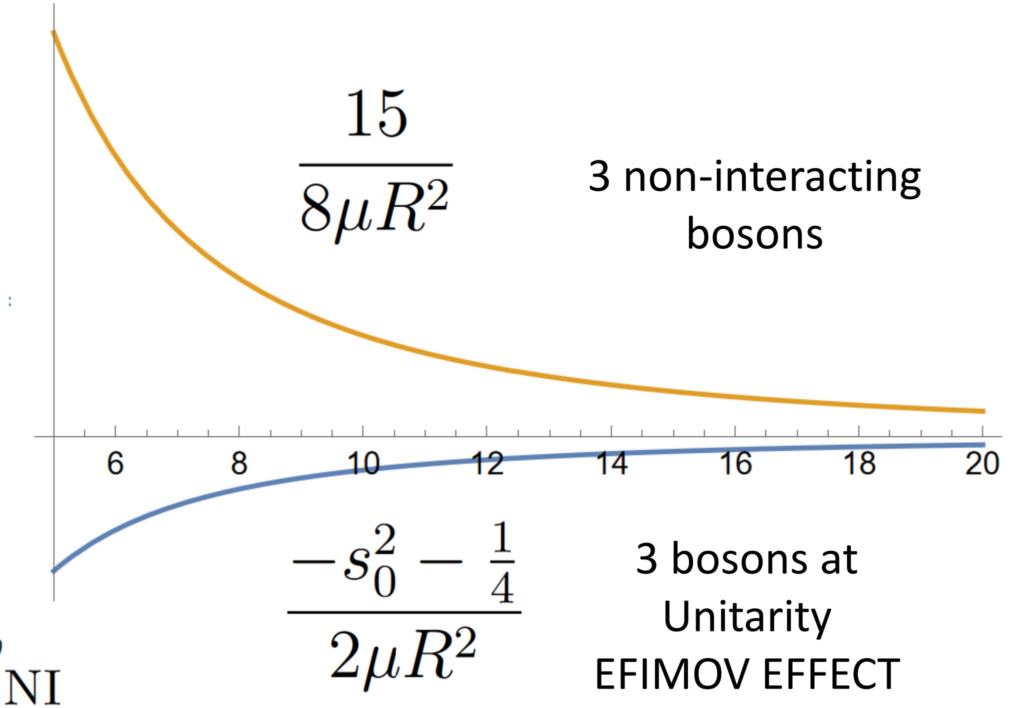
$$W_0^N(R) \rightarrow \frac{\hbar^2}{2\mu}\left[\frac{3(N-2)(3N-4)}{4R^2} + \frac{C_N a_s}{R^3}\right]$$

Now imagine what happens in the unitary limit, as $a \rightarrow -\infty$

$$C_N = \frac{2(N-1)N^{\frac{2N-1}{2N-2}}\Gamma\left[\frac{3}{2}(N-1)\right]}{\sqrt{2\pi}\Gamma\left[\frac{3}{2}(N-2)\right]}$$

Of course the nature of the long range potential must change, and what happens in every case is that the coefficient of $1/R^2$ changes (or in some cases stays the same):

$$U_{\min}^{\text{Unitarity}}(R) \rightarrow \frac{\ell_{\text{eff}}^u(\ell_{\text{eff}}^u+1)\hbar^2}{2\mu R^2}$$



And where $\ell_{\text{eff}}^u(\ell_{\text{eff}}^u+1) \leq \ell_{\text{NI}}(\ell_{\text{NI}}+1)$

e.g. in the Efimov effect, ~ -5/4 < 15/4

Or more precisely,

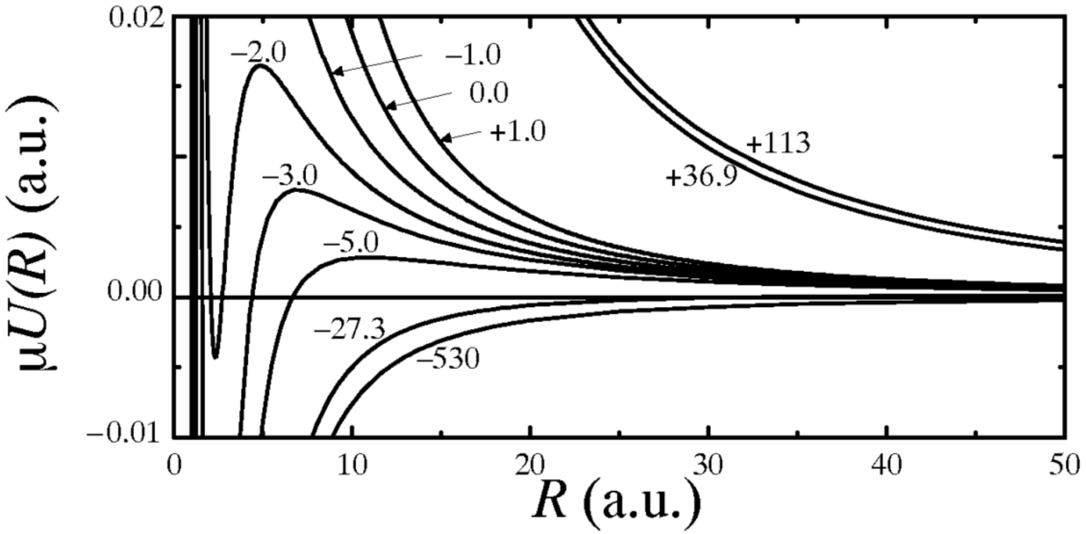
$$\ell_{\text{eff}}^u = -\frac{1}{2} + i s_0 \leftarrow \frac{3}{2} = \ell_{\text{NI}}$$

We sometimes compute the potential curves and couplings ANALYTICALLY, e.g. using zero range potentials and a hyperangular Green's function (Rittenhouse, Mehta, CHG, PRA 82 022706)

But for 3 particles and finite-range potentials, we use a finite-element or B-spline basis set numerical method to solve for $U_n(R)$ and for the nonadiabatic couplings. This results in solution of a 2-dimensional PDE for J=0 states, or for O(J+1) coupled 2-dimensional PDEs for J>0.

Our first study of the 3-boson problem was:
J. Phys. B: At. Mol. Opt. Phys. 29 (1996) L51–L57 (Esry, CHG, Zhou, Lin)

Distances here are in units of the 2-body potential range



Key conclusions:

- Negative a<0 U has a local maximum and a minimum
- Positive a>0 U has a 3-body potential curve that's completely repulsive, and negative energy dimer channel potentials (not shown here)

For the exceptional case $a_{sc} \to \infty$, equation (5) fails, and Efimov's result is appropriate instead [5, 13, 14],

$$U_\nu(R) \to -\frac{0.631}{\mu R^2} + O(R^{-3}) \qquad R \to \infty .$$

At this point, we understood that Efimov's result is not limited to zero-range models!

One tends to think of the Efimov physics reduction of the unitary long range $1/R^2$ potential as a unique case, but in fact this happens at unitarity for S-waves and for P-waves very often in other cases as well. Here are several examples for 3 or 4 fermions:

TABLE II. Long-range coefficients for the three- and four-body equal-mass systems in the lowest few hyperradial channels for different symmetries. The columns specify the symmetry L^π, the hyperspherical channel index ν corresponding to a channel that gives a reduced value of l_{eff} at unitarity, the noninteracting (l_{eff}) and unitary (l_u) angular momentum quantum numbers, and the universal $1/R^3$ coefficient C_ν. The label l_u^{ref} indicates the value of l_u extracted from various references. The error bars are estimated based on convergence of the basis used and on curve fitting at large R.

L^π	ν	l_{eff}	l_u (present)	l_u^{ref}	C_ν	
			(a) (↑↑↓) system			
0^+	1	7/2	1.668(2)	1.6662[a], 1.682[b]	29.47(5)	
	2	11/2	4.628(2)	4.6274[a]	43.37(5)	(↑↑↓) system
1^-	1	5/2	1.273(2)	1.2727[a], 1.275[b]	14.19(5)	
	2	9/2	3.859(2)	3.85825[a]	14.63(5)	
0^+	1	5	2.02(2)	2.028[c]	72.0(3)	(↑↑↓↓) system
	2	7	4.45(5)	4.441[d]	138(2)	

3 fermions, 2 spin components

4 fermions, 2 spin components

D. Blume, J. Von Stecher, and C. H. Greene, Universal Properties of a Trapped Two-Component Fermi Gas at Unitarity, PRL **99**, 233201 (2007).
F.Werner and Y. Castin, Unitary Quantum Three-Body Problem in a Harmonic Trap, PRL **97**, 150401 (2006).

Summary of basic properties of adiabatic hyperspherical potential energy curves:

1. The hyperangular operator Λ^2 is Hermitian and has R-independent eigenvalues, which means that the diagonal potentials $U_n(R)$ for the NONINTERACTING system (V=0) are all known analytically, for all N

$$U_\nu(R) \to \frac{\ell_{\rm NI}(\ell_{\rm NI}+1)\hbar^2}{2\mu R^2}$$

e.g. for 3 bosons, the lowest value of $\ell_{\rm NI}$ is 3/2, whereas for 4 bosons, the lowest value of $\ell_{\rm NI}$ is 3. This is important for discerning threshold laws of transition matrix elements to and/or from the N-body continuum states, as in recombination, i.e. for any short range dominated process,

$$T_{f,i} \propto k_f^{\ell_f+1/2} k_i^{\ell_i+1/2}$$

2. For every possible fragmentation threshold of the N-body system allowed for the symmetry being considered, there must be at least one potential curve converging to its energy at R → infinity

3. When an Efimov effect is present, at least one potential curve converging to E=0 is negative, meaning that

$$\ell_{\rm eff} \equiv -\frac{1}{2} + i s_0$$

4. For s-wave δ-function interactions in 3D with finite *a*, the asymptotic potentials converging to the N-body continuum have the form:

$$\text{as } R \to \infty, U_{\rm min}(R) \to \frac{\ell_{\rm NI}(\ell_{\rm NI}+1)\hbar^2}{2\mu R^2} + C\frac{a}{R^3}$$

This leads us to adopt the following terminology, inspired by the nature of the adiabatic hyperspherical potential curves :

1. "Efimov Physics" is the reduction of the $1/R^2$ long range barrier at unitarity for a channel in the N-body continuum

2. The "Efimov effect" is when that coefficient is more negative than $(-1/8\ \mu)$, resulting in an infinite geometric series of bound or resonant states with decreasing binding energy ➔ log periodicity in E=0 solution, $J_{is0}(kR)$ in general

3. Later we will consider a third category that apparently arises only at P-wave unitarity, the "faux-Efimov effect"

Next, let's look at our application of the hyperspherical picture to the tri-neutron and tetra-neutron systems, and see why it CLEARLY and UNAMBIGUOUSLY demonstrates why there are NO bound or resonant states in either system

Higgins

Kievsky

Viviani

PRL 125, 052501 (2020)

Phys. Rev. C 103, 024004 (2021)

These calculations utilized the correlated Gaussian hyperspherical method (CGHS).

See Daily & CHG, PRA 89, 012503 (2014)

Considerations about the UNITARY limit for 4 fermions

Rakshit and Blume, Phys. Rev. A 86, 062513 (2012) found that as **a-> - infinity**, the 0^+ lowest hyperspherical potential is entirely repulsive, at $|a| >> \rho$:

$$U^{\text{unitary}}(\rho) \rightarrow \frac{6\hbar^2}{2\mu\rho^2} \qquad \ell_{\text{eff}}^U \approx 2.0$$

Whereas in the noninteracting limit the asymptotic potential for two spin-up and two spin-down identical fermions is known to be:

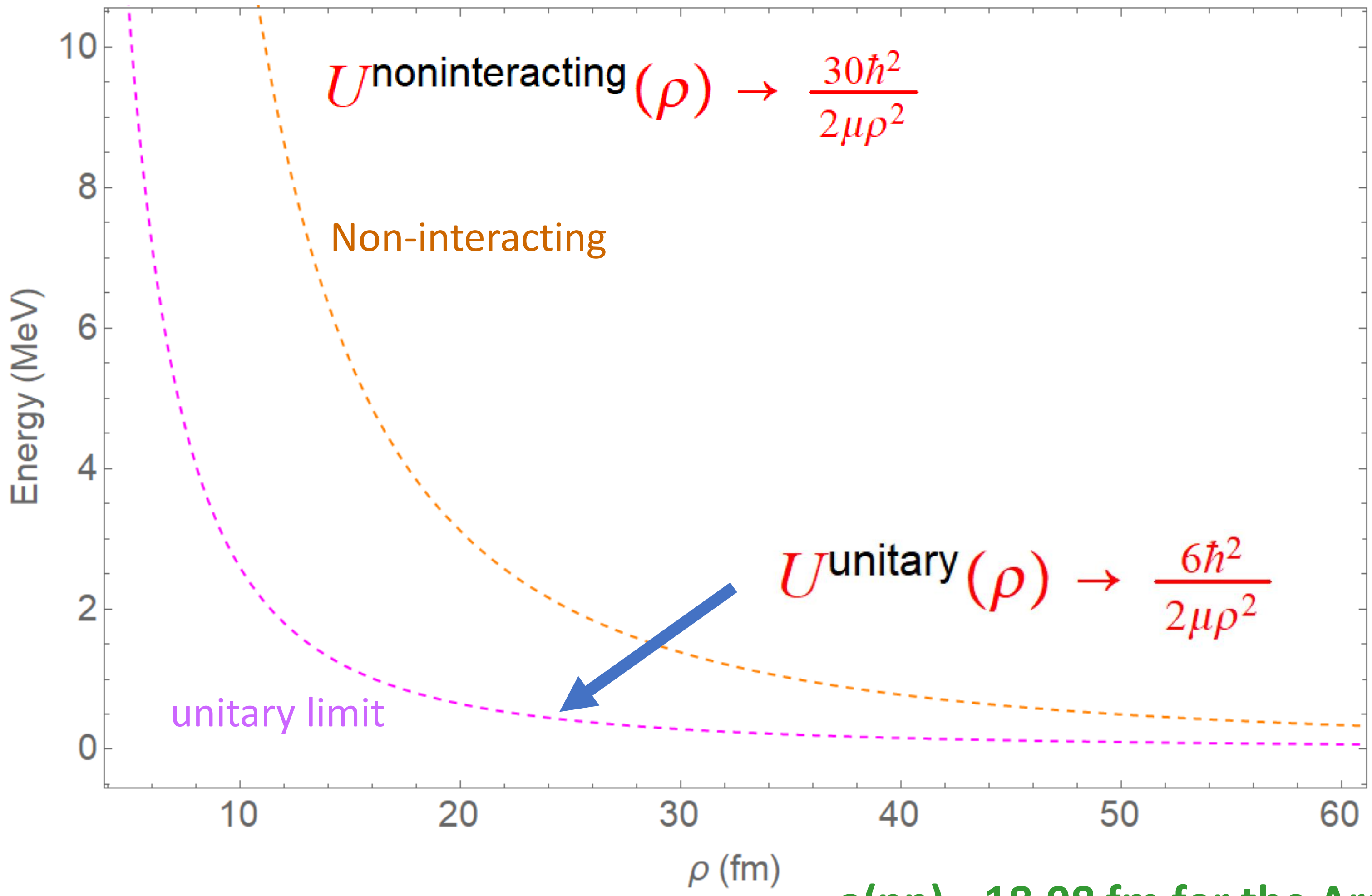
$$U^{\text{noninteracting}}(\rho) \rightarrow \frac{30\hbar^2}{2\mu\rho^2} \qquad \ell_{\text{eff}}^{NI} = 5$$

Aside: This reduction of the coefficient of the asymptotic $1/\rho^2$ potential in the unitary limit, again, is analogous to the Efimov physics...

First Conclusion: The true potential for 4n in this symmetry is expected to be less attractive than the lower of these two potential curves, making the possibility of a bound state for this symmetry unlikely.

a(nn)=-18.98 fm for the Argonne AV8' potential, similar for AV18.

J^{π} =0^+ Hyperspherical potentials for 4 fermions: both noninteracting and unitary limit a→-infinity



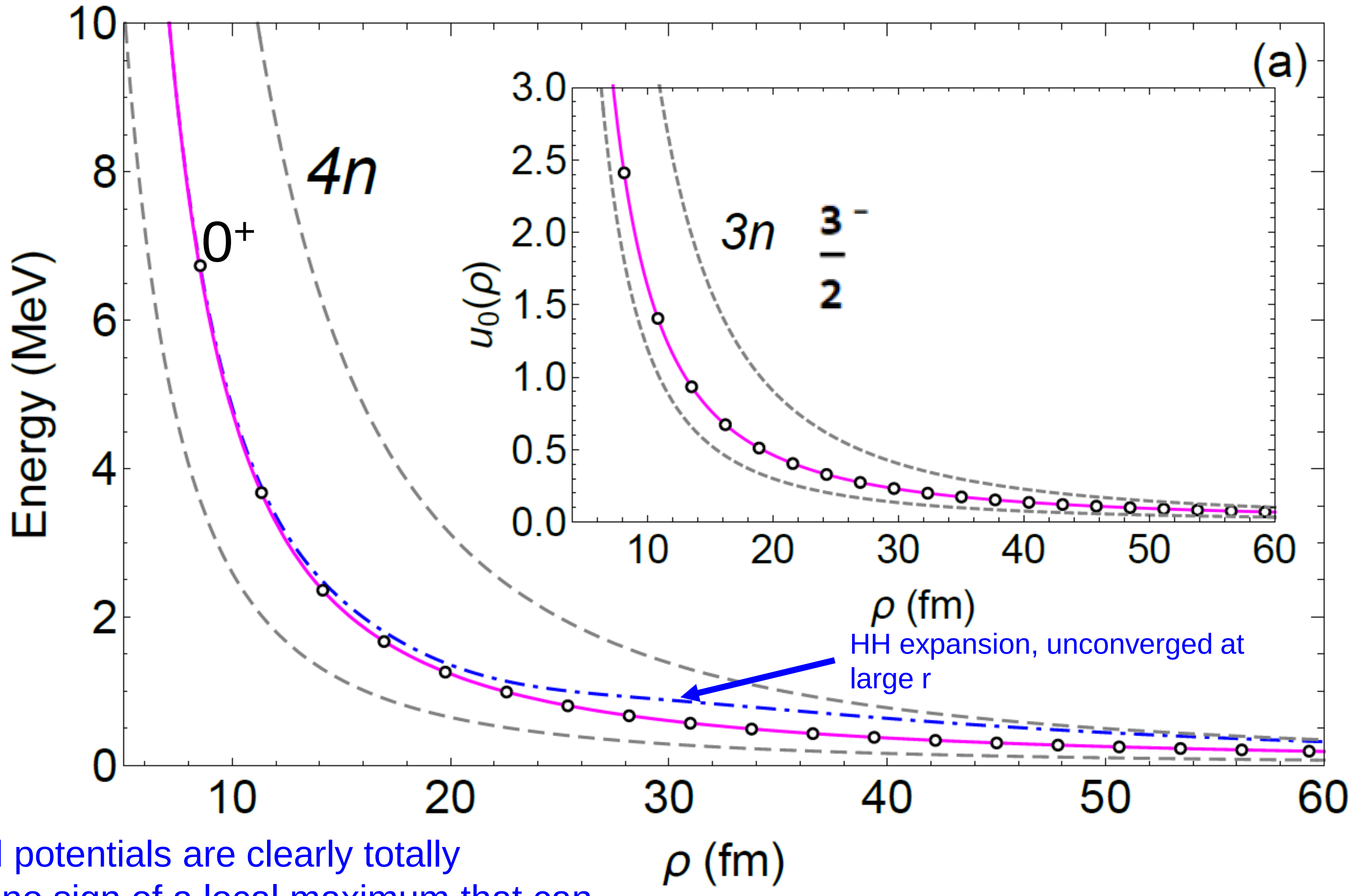
a(nn)=-18.98 fm for the Argonne AV8' potential, similar for AV18.

After failing to get satisfactory convergence at very long range using a hyperspherical harmonic expansion, we decided to implement our stochastic *correlated Gaussian* hyperspherical basis set method CGHS, using the AV8' potential, which had been fitted to Gaussians by Emiko Hiyama and provided to us.

$$\Phi_\nu(R,\Omega) = \hat{S} \sum_j c_j e^{-\frac{1}{2}x^T A_j x} [\Theta_L^K(u,x) \times \chi_S]_{JM_J}$$

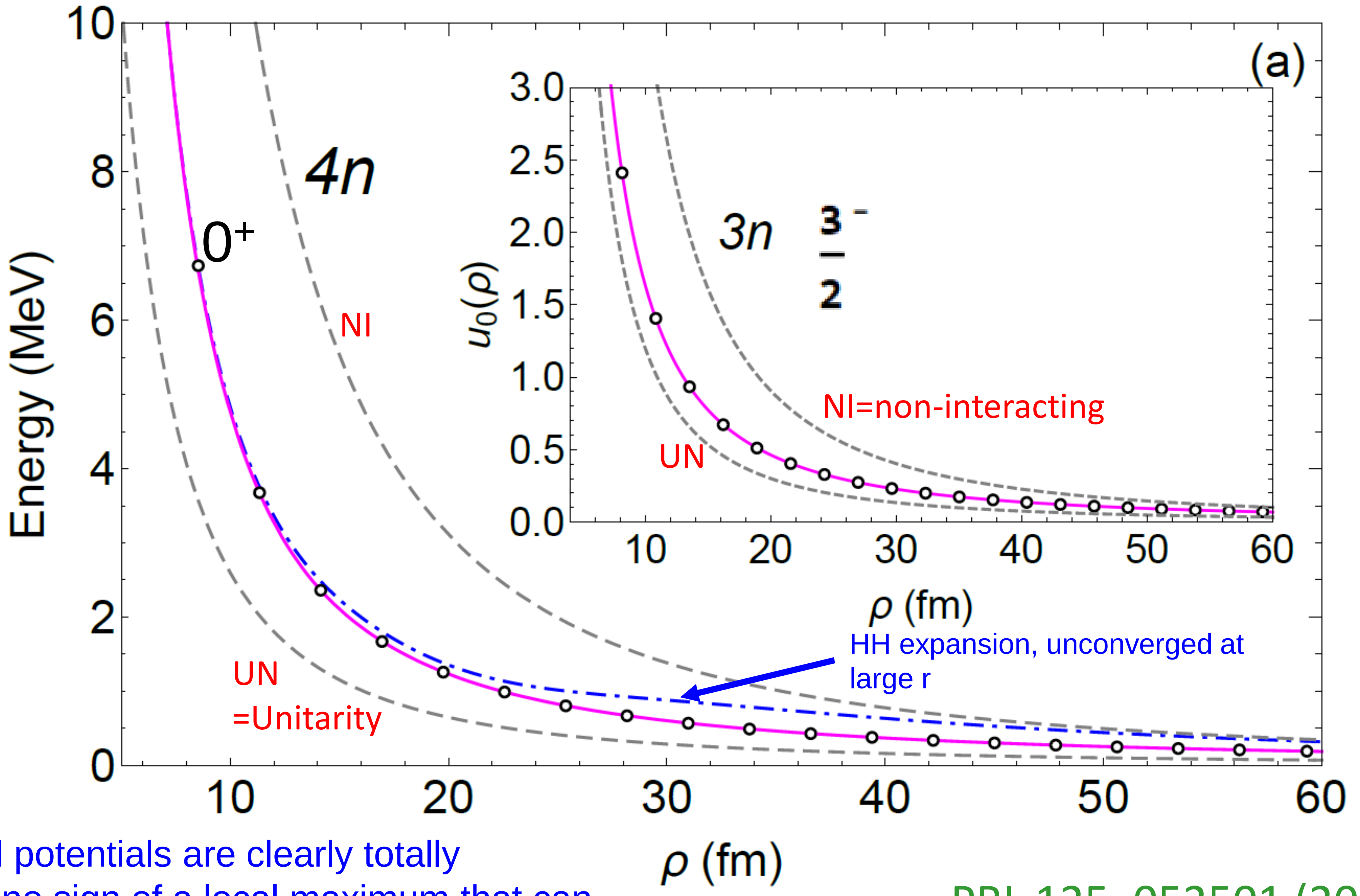
Side note: in order to be thorough, we also treated other nuclear force models, i.e. AV8', AV18, Minnesota potential, a chiral EFT potential (NV2-Ia) , and with or without the 3-body n-n-n term from the Urbana and Illinois (e.g. IL) interaction. Minimal differences are observed with these different interactions

The most attractive hyperspherical potential curves for the 4n and 3n systems, obtained using the AV8' n-n interaction potentials (magenta)



The converged potentials are clearly totally <u>repulsive</u>, with <u>no sign of a local maximum</u> that can trap probability in a resonance.

The most attractive hyperspherical potential curves for the 4n and 3n systems, obtained using the AV8' n-n interaction potentials (magenta)



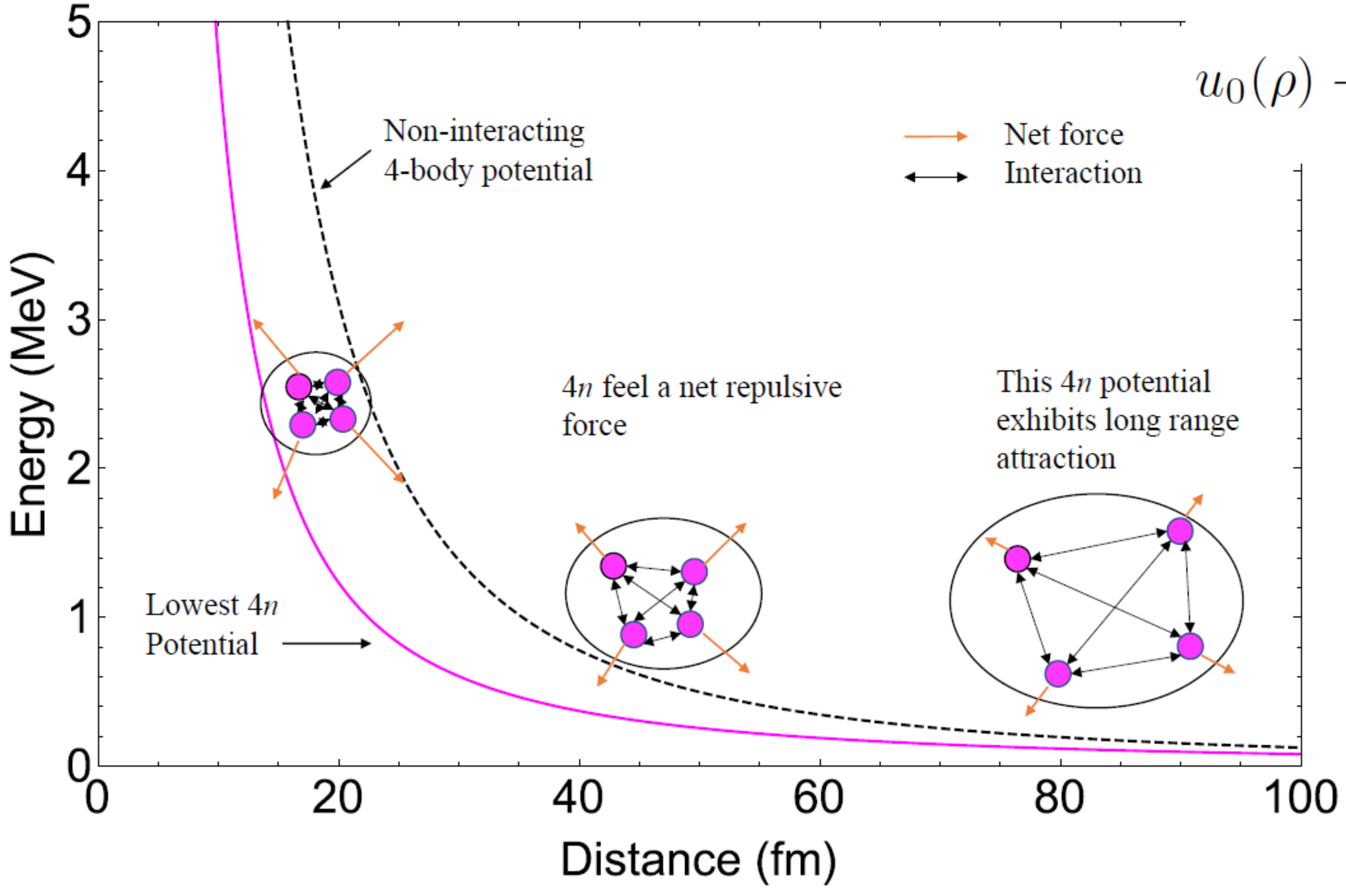
The converged potentials are clearly totally repulsive, with no sign of a local maximum that can trap probability in a resonance.

PRL 125, 052501 (2020)

Main conclusions about the tetra-neutron and tri-neutron study:

(a) There is strong attraction in the system at each hyperradius that lowers the potential energy, associated with a(nn) ~ -19 fm (in the singlet channel)

(b) The attraction diminishes with increasing hyperradius, such that the tetraneutron always experiences an outward force, as shown in the cartoon



$$u_0(\rho) \to \frac{\hbar^2}{2\mu}\left(\frac{l_{\rm eff}(l_{\rm eff}+1)}{\rho^2} + C\frac{a}{\rho^3}\right)$$

Let’s talk about threshold laws and their modifications near unitarity

Reference: J. Phys. B: At. Mol. Opt. Phys. **33** (2000) R93–R140
H R Sadeghpour, J L Bohn, M J Cavagnero, B D Esry, I I Fabrikant, J H Macek and A R P Rau

Point #1: The true threshold behavior for a process is controlled by the longest range potential in that channel which is approaching that channel’s threshold energy, and if that potential varies asymptotically as a repulsive potential of the form

$$\frac{\ell_\nu(\ell_\nu + 1)}{2\mu R^2}$$

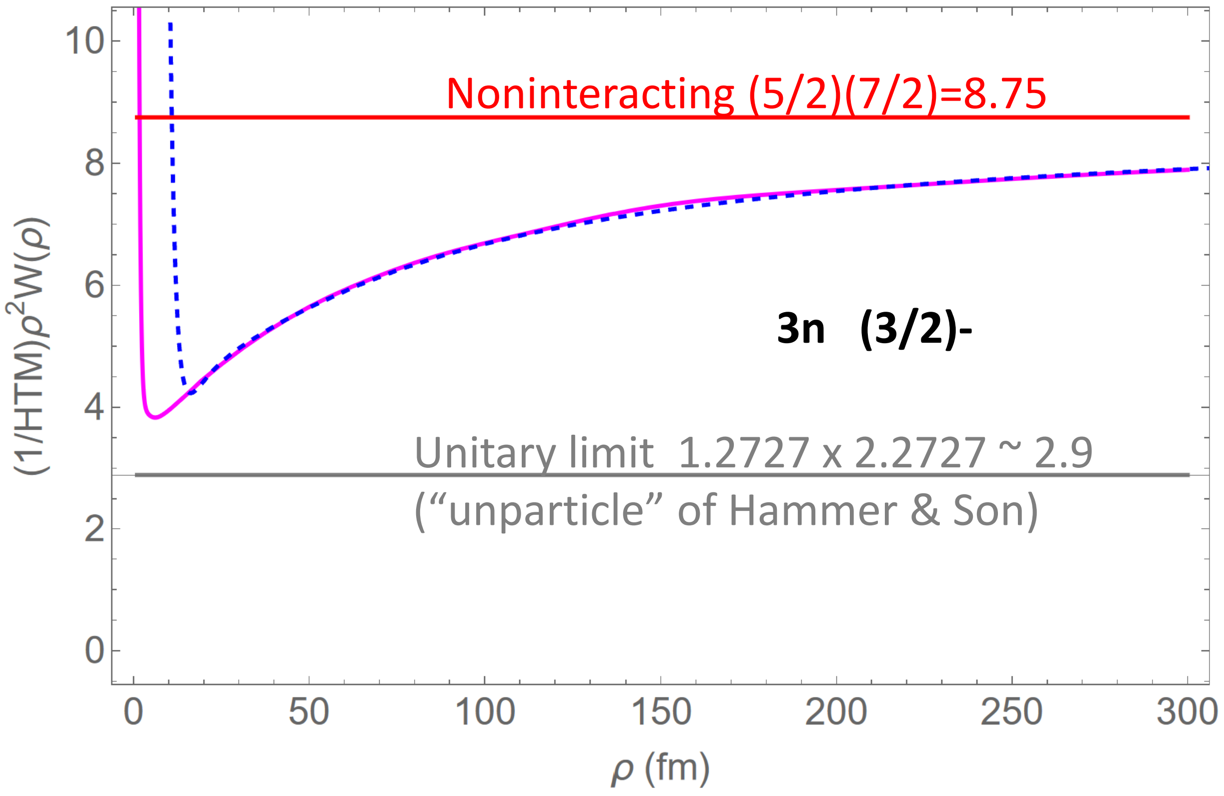
Then the threshold behavior associated with that channel for any collision process entering or exiting at an energy just above threshold will be the <u>Wigner threshold law</u> for that value of ℓ_ν namely:

$$S_{fi} \propto k^{\ell_\nu + 1/2}$$

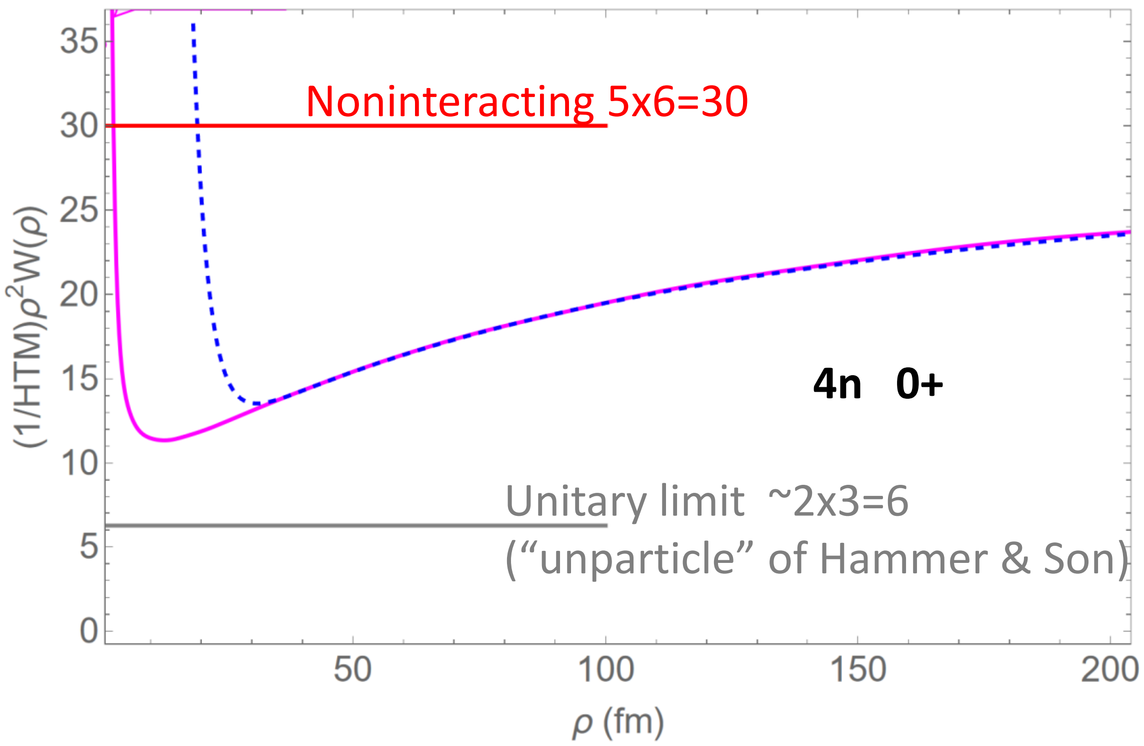
Fit to the asymptotic 3n and 4n potentials, multiplied by $2m\,\rho^2/\text{hbar}^2$

This is a study of the 3n,4n long range attractions, showing a very slow convergence of the potentials

8.75 + b/ρ + c/ρ^2 + d/ρ^3 + f/ρ^4 is the blue dashed fit below



30 + b/ρ + c/ρ^2 + d/ρ^3 + f/ρ^4 is the blue dashed fit



Next, let's investigate the energy dependent probability of the 3n and 4n systems to reach small hyperradii, which gives the energy dependence relevant to any matrix element of an operator that produces 3n or 4n via a short-range process

I will now argue that this analysis can be carried out accurately using WKB to calculate the tunneling under the centrifugal barrier down to short distances, as a generalization of the Wigner threshold law

(we also carried out the calculations by directly solving the radial Sch. Eqn and obtain the same results, but I will give here the simpler WKB-based analysis which works perfectly)

Aside on the interpretation of Wigner threshold law factors as a tunneling integral

Here is a brief description about how one can understand the Wigner threshold law for any transition matrix element that is controlled by the short-range part of the near-threshold wavefunction. Consider a general potential with the following asymptotic form:

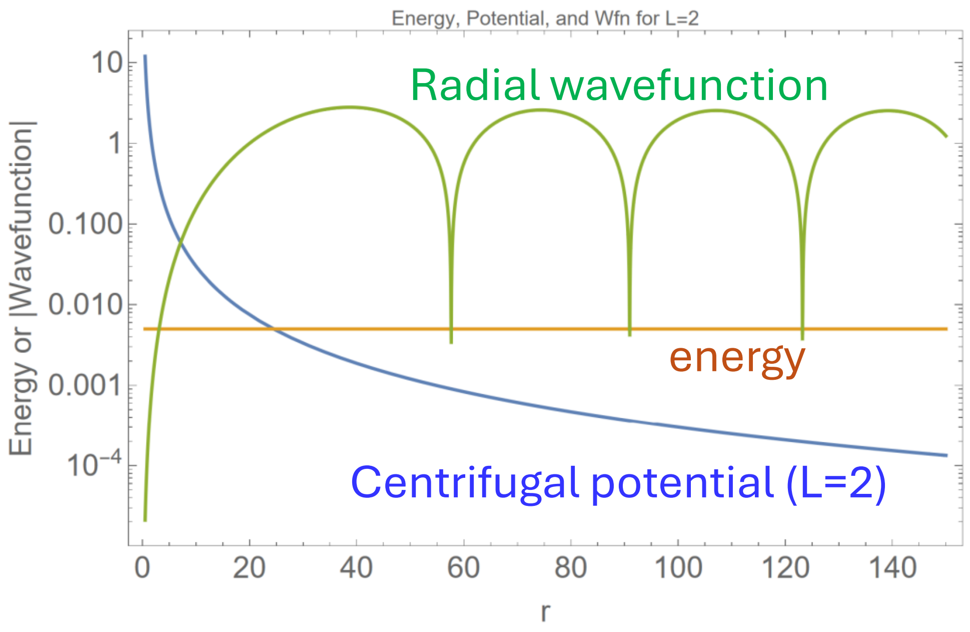
$$V(\rho) \to \frac{\ell(\ell+1)\hbar^2}{2\mu\rho^2}, \text{as} \quad \rho \to \infty$$

It is not often taught in textbooks, but the Wigner threshold law energy-dependent factor in the transition matrix element can be viewed as the tunneling amplitude through the centrifugal barrier. To see this, consider the WKB tunneling integral from the long range turning point at $\rho \approx \frac{\ell+\frac{1}{2}}{k}$ into a short distance:

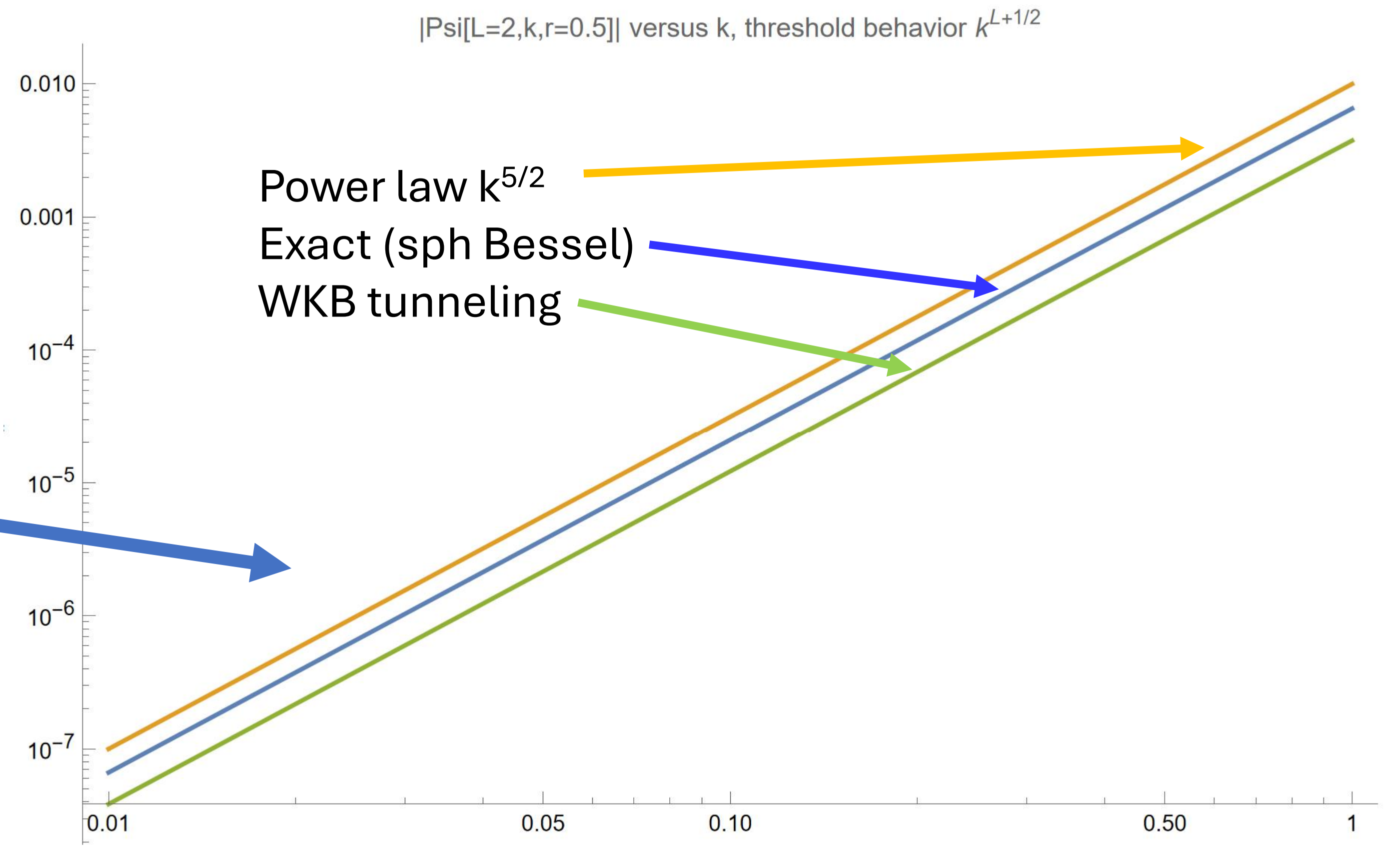
$$\psi(k, \rho_{\text{small}}) \propto \exp\left(- \int_{\rho_{\text{small}}}^{\frac{\ell+\frac{1}{2}}{k}} Re\sqrt{\frac{(\ell+\frac{1}{2})^2}{\rho'^2} - k^2 d\rho'} \right)$$

$$\approx \exp\left(-(\ell + \frac{1}{2}) ln(k\rho_{\text{small}}) \right) \propto k^{\ell+\frac{1}{2}}$$

Demonstration that this WKB argument gives the correct Wigner threshold law for an L=2 energy-normalized wavefunction



For this d-wave example, we confirm that both the exact spherical Bessel solution AND the WKB tunneling calculation agree on the Wigner threshold law energy dependence at low energy, namely Psi ~ $k^{(L+1/2)}$



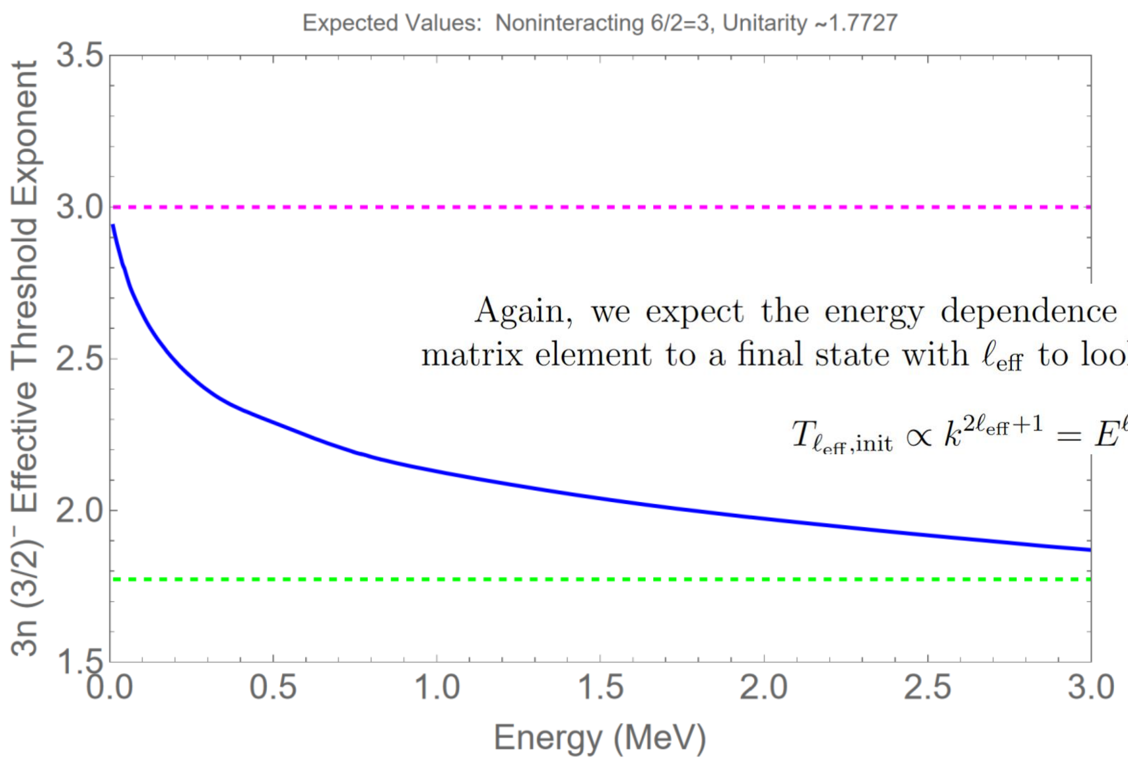
Checking the expected threshold behavior of a transition matrix element involving the **3-neutron 1⁻ final state wavefunction** as a function of the relative energy

Note that if the final state wavefunction at short range is controlled by the outermost classical turning point at $R \approx \frac{\ell_{\rm eff}+1/2}{k}$ and if that turning point varies slowly with k, then we can define an effective energy-dependent threshold law exponent γ in $\psi \propto k^{\gamma}$, using the formula:

$$\gamma(k) \approx k \frac{d}{dk} \ln\left(\psi(k, \rho_{\rm small})\right)$$

Analysis is based on the following tunneling integral with our 3n hyperspherical potential curves

$$\psi(k, \rho_{\rm small}) \propto \exp\left(- \int_{\rho_{\rm small}}^{\frac{\ell+\frac{1}{2}}{k}} Re \sqrt{\frac{2\mu U(\rho)}{\hbar^2} + \frac{(\frac{1}{2})^2}{\rho'^2} - k^2 d\rho'} \right)$$



←expected threshold exponent 3 very close to threshold (non-interacting exponent)

Again, we expect the energy dependence of any short range transition matrix element to a final state with $\ell_{\rm eff}$ to look like:

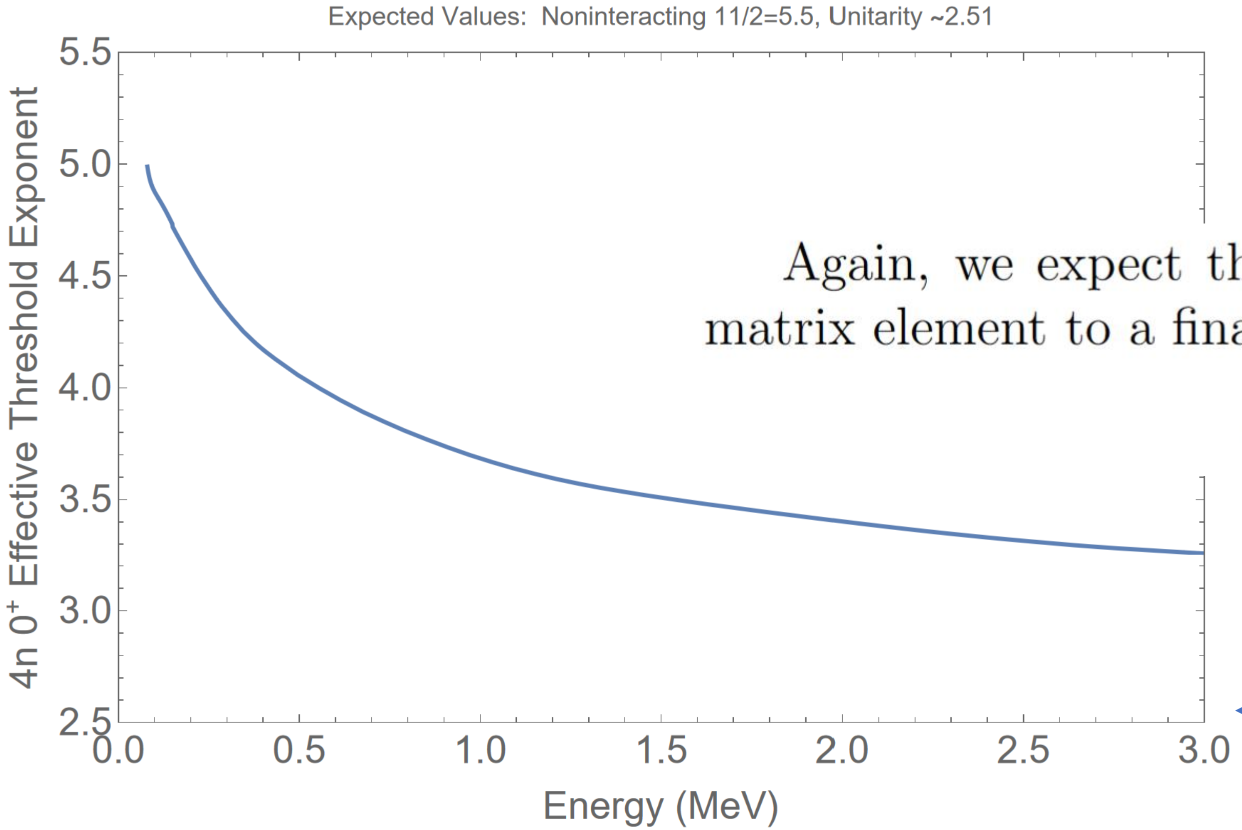
$$T_{\ell_{\rm eff},{\rm init}} \propto k^{2\ell_{\rm eff}+1} = E^{\ell_{\rm eff}+1/2}$$

← expected unitary exponent ~1.77 very close to threshold (unitarity limit, Hammer & Son PNAS suggest that ~1.77 should be the observable threshold exponent for the 3n system in 1- symmetry)

Similarly can get the expected threshold behavior of a transition matrix element involving the **4-neutron 0+ final state wavefunction** as a function of the relative energy

Note that if the final state wavefunction at short range is controlled by the outermost classical turning point at $R \approx \frac{\ell_{\text{eff}}+1/2}{k}$ and if that turning point varies slowly with k, then we can define an effective energy-dependent threshold law exponent γ in $\psi \propto k^{\gamma}$, using the formula:

$$\gamma(k) \approx k \frac{d}{dk} \ln\left(\psi(k, \rho_{\text{small}})\right)$$



←expected threshold exponent 5.5 very close to threshold (non-interacting exponent)

Again, we expect the energy dependence of any short range transition matrix element to a final state with ℓ_{eff} to look like:

$$T_{\ell_{\text{eff}},\text{init}} \propto k^{2\ell_{\text{eff}}+1} = E^{\ell_{\text{eff}}+1/2}$$

expected unitary exponent ~2.51 very close to threshold (unitarity limit, Hammer & Son PNAS state that ~2.5 to 2.6 should be the observable threshold exponent)

A theoretical study of low energy processes that produce three neutrons in the 3-particle continuum

Idea of Hammer & Son, PNAS 2021: They can <u>test the unitarity-modified threshold law</u> by looking at a detailed theoretical calculation that produces 3 low energy neutrons in the continuum, from 2 papers:

J. Golak *et al.*, Radiative pion capture in ^{2}H, ^{3}He and ^{3}H. *Phys. Rev. C* **98**, 054001 (2018).

J. Golak *et al.*, Muon capture on ^{3}H. *Phys. Rev. C* **94**, 034002 (2016).

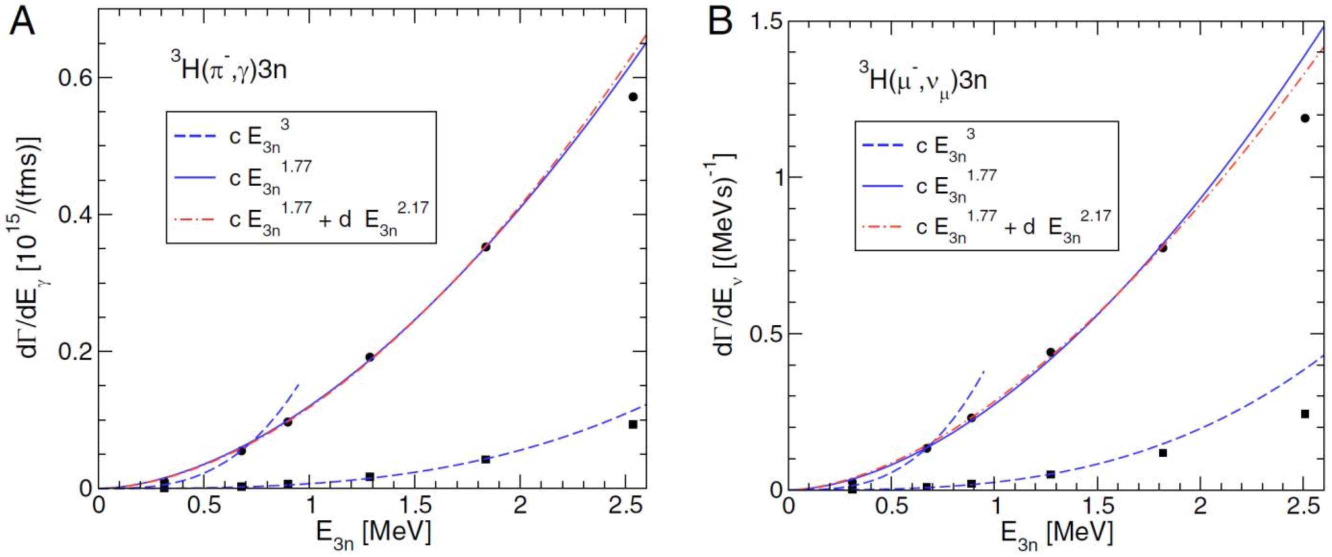
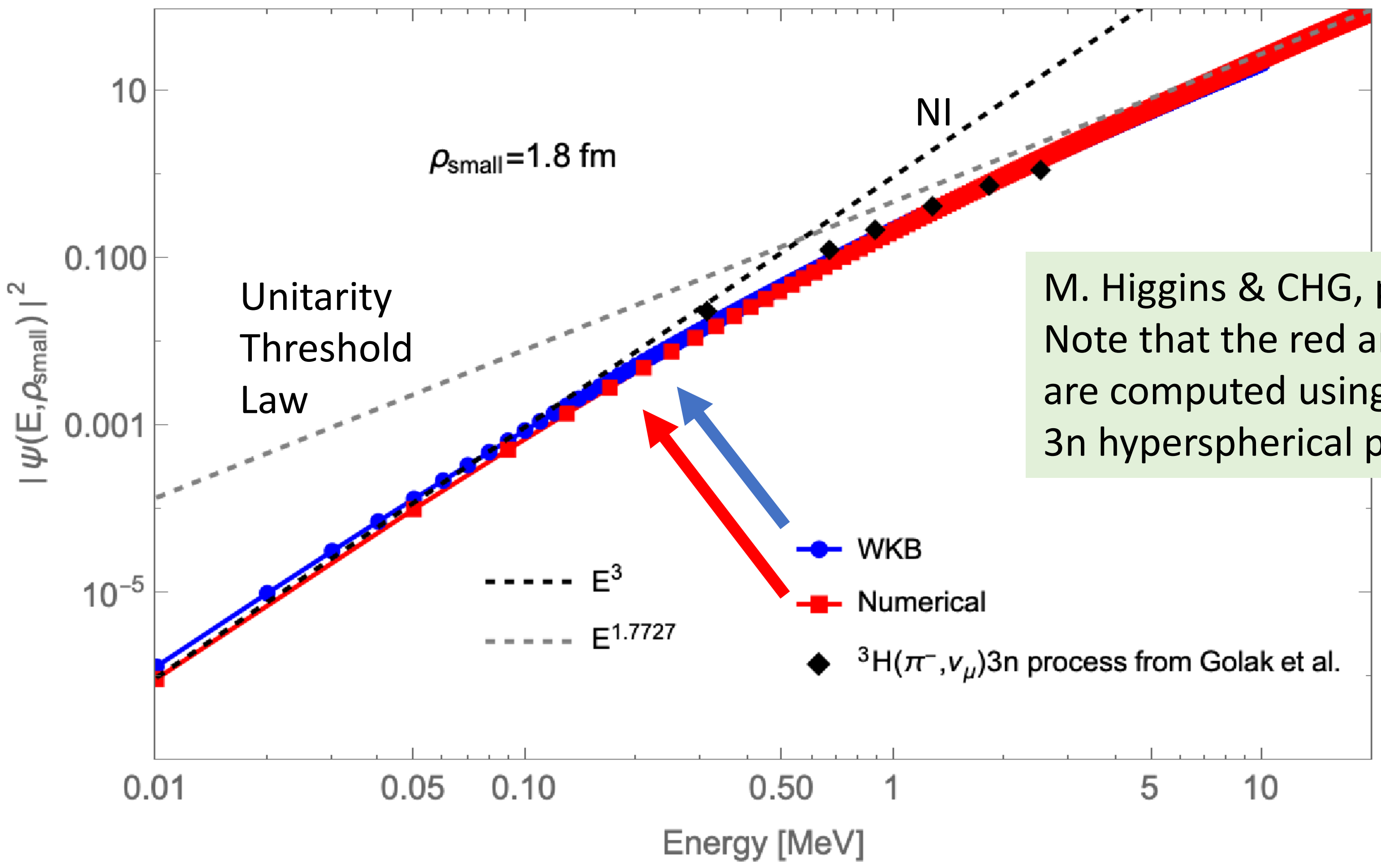


Fig. 5. Center-of-mass energy spectrum of three neutrons in the reaction $^3H(\pi^-,\gamma)3n$ (*A*) and $^3H(\mu^-,\nu_\mu)3n$ (*B*). The circles/squares give the full/plane wave calculations by Golak et al. (23, 24). Different fits are explained in the key and in the main text.

On this scale, the agreement with the unitarity-modified threshold exponent appears to be pretty satisfactory. BUT, let's take a more detailed study by plotting these results on a log scale



M. Higgins & CHG, preliminary
Note that the red and blue points are computed using our lowest 3n hyperspherical potential curve

Even though the data are relatively sparse here at low energy (6 calculated points from Golak et al), the energy dependence of the short range wavefunction is better described by the hyperspherical potential curve prediction, and it shows the expected (NI) Wigner threshold law below about 0.15 MeV. So even the 3n system is "<u>not THAT close to unitarity</u>"

On p-wave universality and the faux-Efimov effect: (note: “faux” means “false”)

Background:

(1) For single component fermionic trimers, a P-wave Efimov effect in the symmetry $L^{\pi}=1^{-}$ was initially predicted by Macek and Sternberg (2006 PRL), but the prediction was shown by Braaten & Hammer and by Nishida (2012 PRA) because the states computed by Macek and Sternberg had negative probability and were not normalizable.

(2) Another prediction of a fermionic P-wave Efimov effect was made by Braaten, Hagen, Hammer, and Platter, in an article titled: *Renormalization in the three-body problem with resonant p-wave interactions*, PRA 86, 012711; this prediction of an Efimov effect for $L^{\Pi} = 0^{+}, 1^{\pm}$ and 2^{+} was also shown to be incorrect by Nishida for essentially the same reason.

CONCLUSION: there is something interesting and unusual that resulted in these predictions, so let’s see how the phenomenon can be explained within the adiabatic hyperspherical viewpoint

First of all, here is the terminology we use:

$$\left[-\frac{\hbar^2}{2\mu_{3b}}\frac{d^2}{dR^2}+W_\nu(R)-E\right]F_\nu^E(R)$$
$$+\sum_{\nu'\neq\nu}\left(-\frac{\hbar^2}{2\mu_{3b}}\right)\left[2P_{\nu\nu'}(R)\frac{d}{dR}+Q_{\nu\nu'}(R)\right]F_{\nu'}^E(R)=0 \quad (10)$$

In the above expression, E is the total energy and $W_\nu(R)$ is the effective *adiabatic potential* in channel ν:

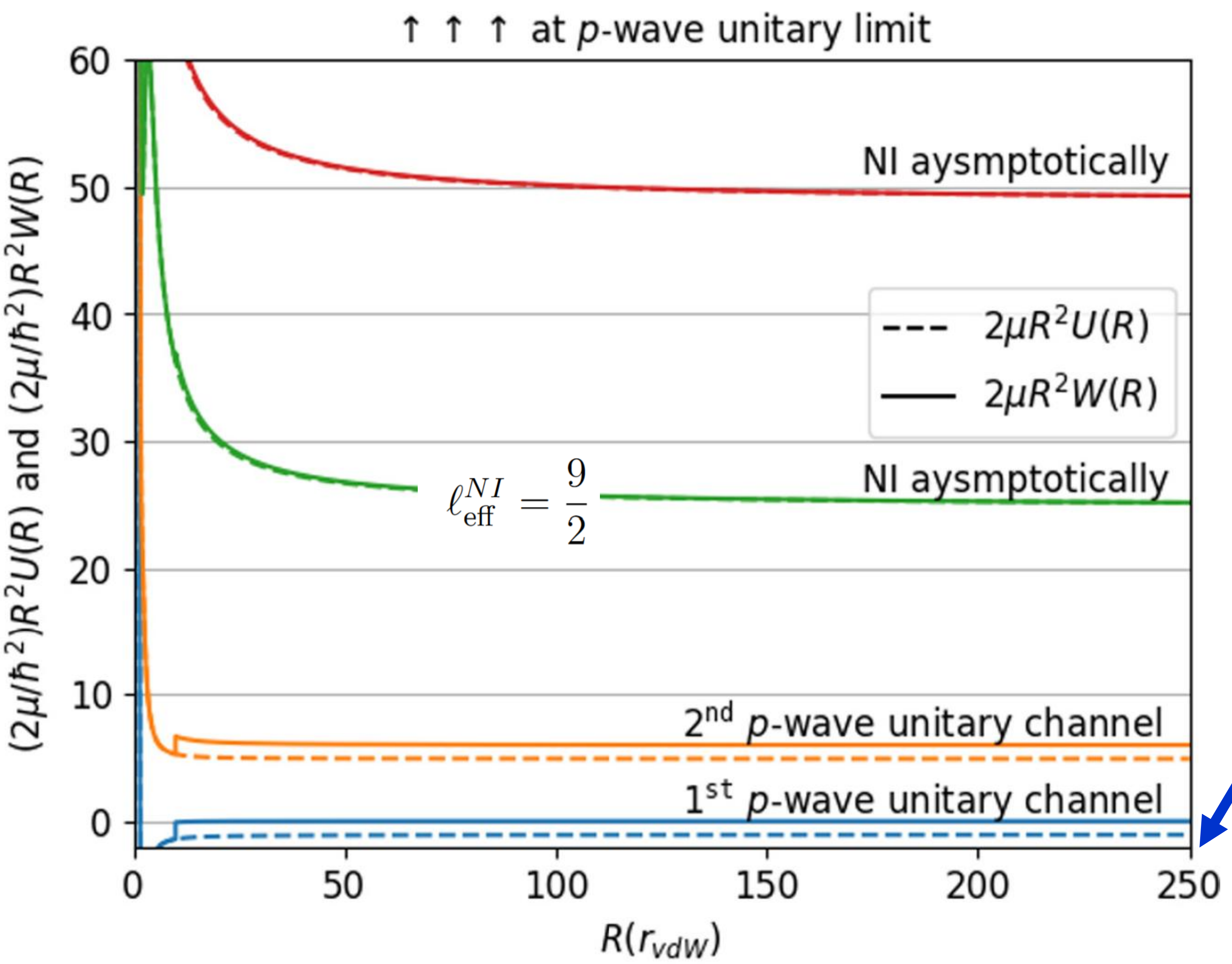
$$W_\nu(R)\equiv U_\nu(R)-\frac{\hbar^2}{2\mu_{3b}}Q_{\nu\nu}(R). \quad (11)$$

The adiabatic potential includes the so-called 'diagonal correction' or 'Born-Huang correction' $Q_{\nu\nu}(R)$, while we sometimes refer instead to the uncorrected 'Born-Oppenheimer' potential curve $U_\nu(R)$.

Note that the Born-Oppenheimer version neglecting Q(diagonal) is sometimes studied because it gives a LOWER BOUND.

But in this context it is dangerous to omit it.

Here we multiply U or W by R^2 to look at the asymptotic coefficient of $1/R^2$.



Example of a case with a faux-Efimov effect in a spin-polarized fermionic trimer.

Here $L^\pi = 1^-$

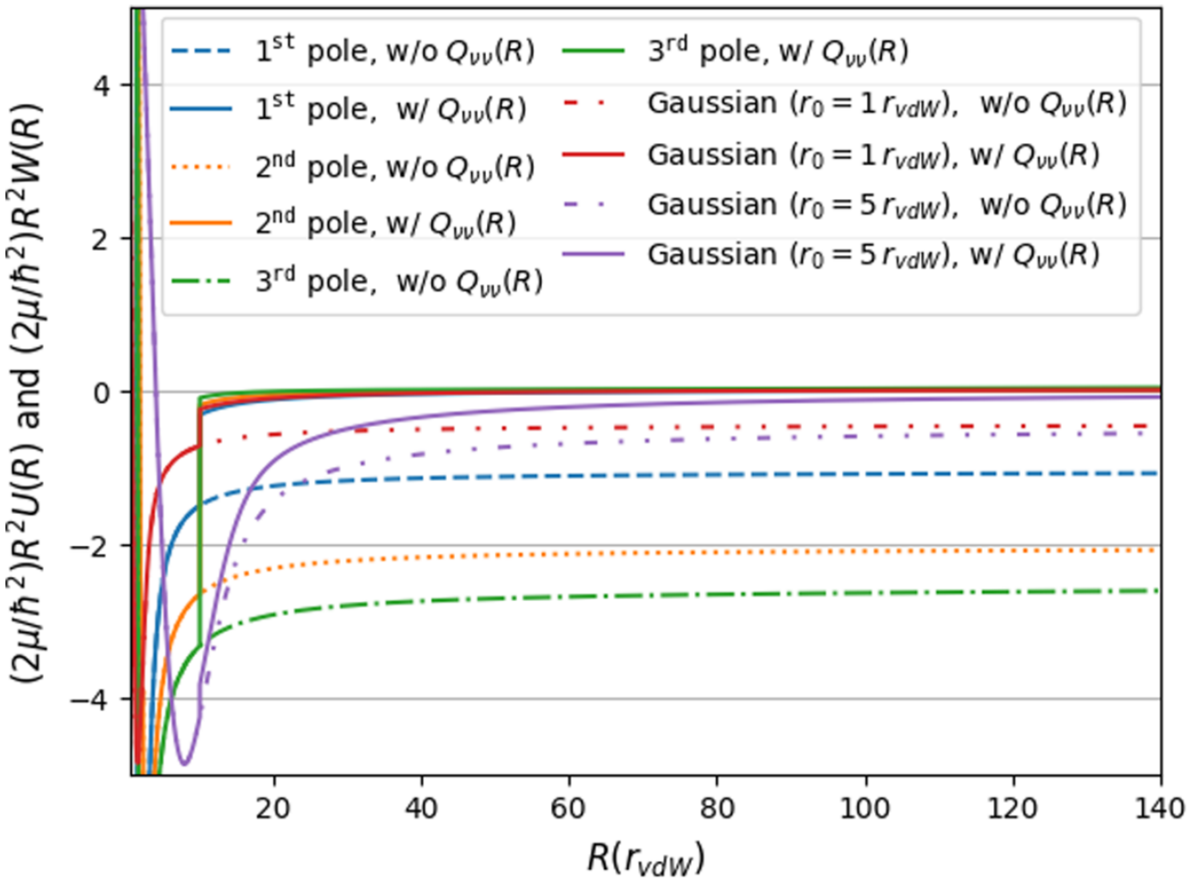
The green and red rescaled potentials are obtained for zero 2-body scattering volume, and they are highly repulsive.

The blue and orange channels potentials are completely new when the 2-body scattering volume is infinite.

The dashed curves are Born-Oppenheimer and the lowest blue dashed curve is "faux-Efimov". When the diagonal correction is added (solid), the resulting (physical) adiabatic potential has coefficient very close to $0/R^2$ and no Efimov effect

Illustration of the non-universal nature of the (unphysical) faux-Efimov Born-Oppenheimer potentials (rescaled) at R → infinity, i.e. by considering different p-wave poles with different numbers of two-body bound states, but the universal (physical) potentials nevertheless collapse to $0/R^2$

YU-HSIN CHEN AND CHRIS H. GREENE



<u>Side note</u>: The system of 4 spin-polarized fermions with unitary p-wave interactions also has a faux-Efimov effect in the 0^+ symmetry. See Higgins & CHG, PRA 2022

$$J^{\Pi} = 1^-$$

This figure is for the two component fermionic trimer at p-wave unitarity for same spins, and s-wave unitarity between unlike spins

L^Π	case	$l_{e,\nu}$								
0^+	non-interacting	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	23/2
	↑↓ at s-wave unitary limit and ↑↑ non-interacting	**1.666**	**4.627**	**6.614**	15/2	**8.332**	19/2	**10.562**	23/2	23/2
	↑↑ at p-wave unitary limit and ↑↓ non-interacting	1	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2
	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit	1	**1.666**	**4.627**	**6.614**	15/2	**8.332**	19/2	**10.562**	23/2
1^+	non-interacting	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	23/2
	↑↓ at s-wave unitary limit and ↑↑ non-interacting	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2	23/2
	↑↑ at p-wave unitary limit and ↑↓ non-interacting	1	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2
	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit	1	7/2	11/2	15/2	15/2	19/2	19/2	23/2	23/2
1^-	non-interacting	5/2	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2
	↑↓ at s-wave unitary limit and ↑↑ non-interacting	**1.272**	**3.858**	9/2	**5.216**	13/2	13/2	**7.553**	17/2	17/2
	↑↑ at p-wave unitary limit and ↑↓ non-interacting	0	2	5/2	9/2	9/2	13/2	13/2	13/2	17/2
	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit	0	**1.272**	2	**3.858**	9/2	**5.216**	13/2	13/2	**7.553**
2^-	non-interacting	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	21/2
	↑↓ at s-wave unitary limit and ↑↑ non-interacting	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2	21/2
	↑↑ at p-wave unitary limit and ↑↓ non-interacting	2	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2
	↑↓ at s-wave unitarity and ↑↑ at p-wave unitary limit	2	9/2	9/2	13/2	13/2	13/2	17/2	17/2	17/2

TABLE II. Comparison of the 1st to 9th $l_{e,\nu}$ values for trimers consisting of two spin-up and one spin-down fermion (↓↑↑) with the non-interacting values of $l_{e,\nu}$. Cases considered involve different spin fermions that are either at s-wave unitarity or else are noninteracting as are labeled in the Table. Other cases show the same spin fermions either at p-wave unitarity or noninteracting. The bold and underlined numbers correspond to the $l_{e,\nu}$ values of the s-wave unitary channel and the p-wave unitary channel, respectively.

Unitarity reductions of the centrifugal barrier constant l_e for Fermionic 3-body systems of different symmetries

Yu-Hsin Chen & CHG
arXiv:2408.08993
Universal trimers with p-wave interactions and the faux-Efimov effect

See also 2022 PRA

The corrected values of l'_ν In the faux-Efimov cases appear to always equal integers within our numerical accuracy level

Aside: beyond the recombination of 3 identical bosons → 3 spin polarized fermions!

Recombination of Three Ultracold Fermionic Atoms
H. Suno, B. D. Esry, and CHG -- PRL 90, 053202 (2003),
appeared back-to-back in PRL with an experimental paper from Deborah Jin's group:
Regal et al., PRL 90, 053201 (2003)

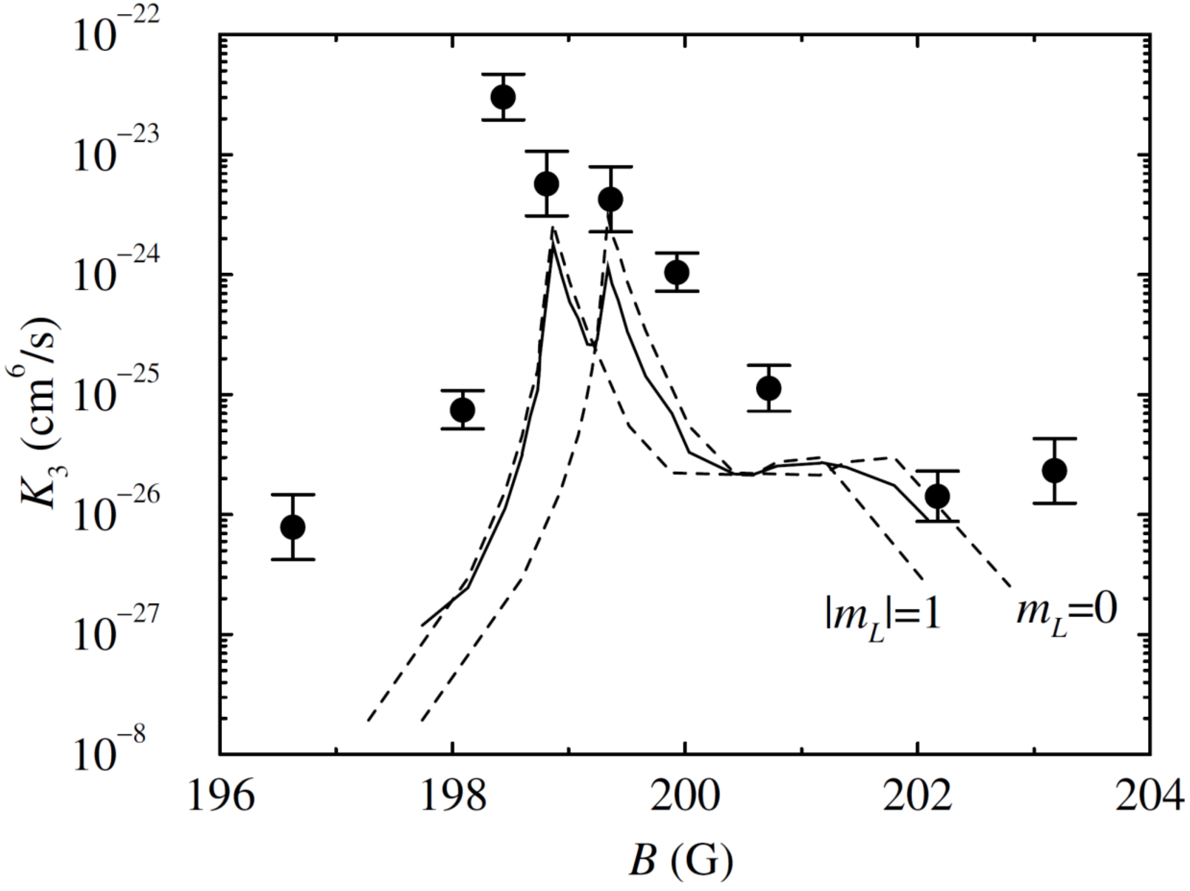
Motivation for this study:
The hope was that by making the fermionic atoms spin polarized, the Pauli antisymmetrization should lead to a huge suppression of 3-body recombination in the ultracold degenerate Fermi gas.

Key results: The Pauli antisymmetrization changes the lowest value of l_{eff} → 7/2 in the 3-body continuum whereas it was l_{eff} = 3/2 for bosons, and the Wigner threshold law for the recombination matrix element must scale with energy as

$$S_{b,fff} \sim k^{2l_{eff}+1} \rightarrow (k\, Vp^{1/3})^{2l_{eff}+1}$$

Where Vp is the p-wave scattering volume and $Vp^{1/3}$ is called the ``p-wave scattering length". Thus the recombination rate should scale with energy as $K_3 \sim E^2$

3-body recombination rate coefficient for spin-polarized fermionic 40K atoms, experiment-theory comparison, experiment from Regal and Jin, 2003 PRL



H. Suno, B. D. Esry, and CHG -- PRL 90, 053202 (2003), appeared back-to-back in PRL with an experimental paper from Deborah Jin's group:
Regal et al., PRL 90, 053201 (2003)

To our surprise, and to the experimentalists' disappointment, the recombination rate still can approach the unitarity limit to within about 30-50%, despite the Pauli exclusion principle.

A first study of 5-body nuclear physics: the α+4n system, ^{8}He
arXiv:2407.17668 by Higgins & CHG

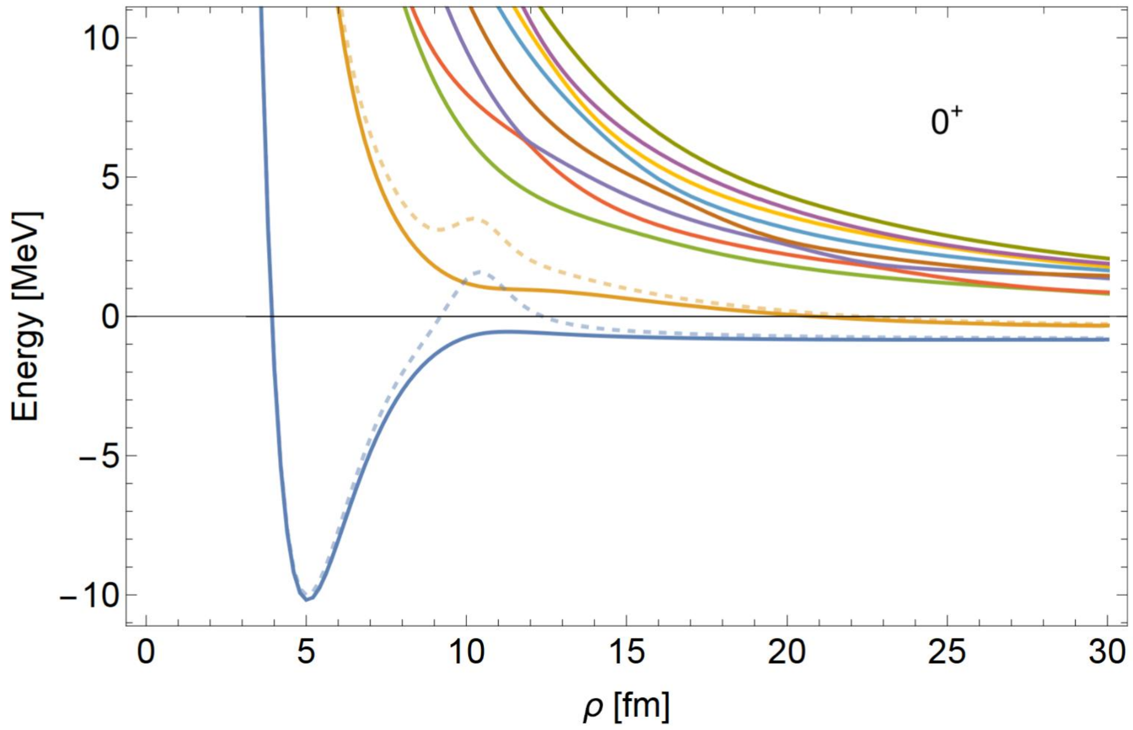


FIG. 9. The lowest few Born–Oppenheimer potential curves for the ^{4}He + 4n system in the $(L^\pi, S)J^\pi = (0,0)0^+$ symmetry. The solid curves are the Born–Oppenheimer potentials and the dashed curves are the adiabatic potentials, which includes the diagonal second–derivative coupling term. Only the lowest two adiabatic potentials are shown here.

One interesting thing about this system that we have not seen before is that there are no true 2-body fragmentation channels at low energy. The lowest channels converge asymptotically to ^{6}He+n+n, And there are an infinity of such potentials

Another challenge: 5-body recombination for 5 free bosonic atoms with pairwise additive forces

i.e. the reaction $A+A+A+A+A \rightarrow A_3+A_2$ or A_4+A or…

Start with the time-independent Schroedinger equation:

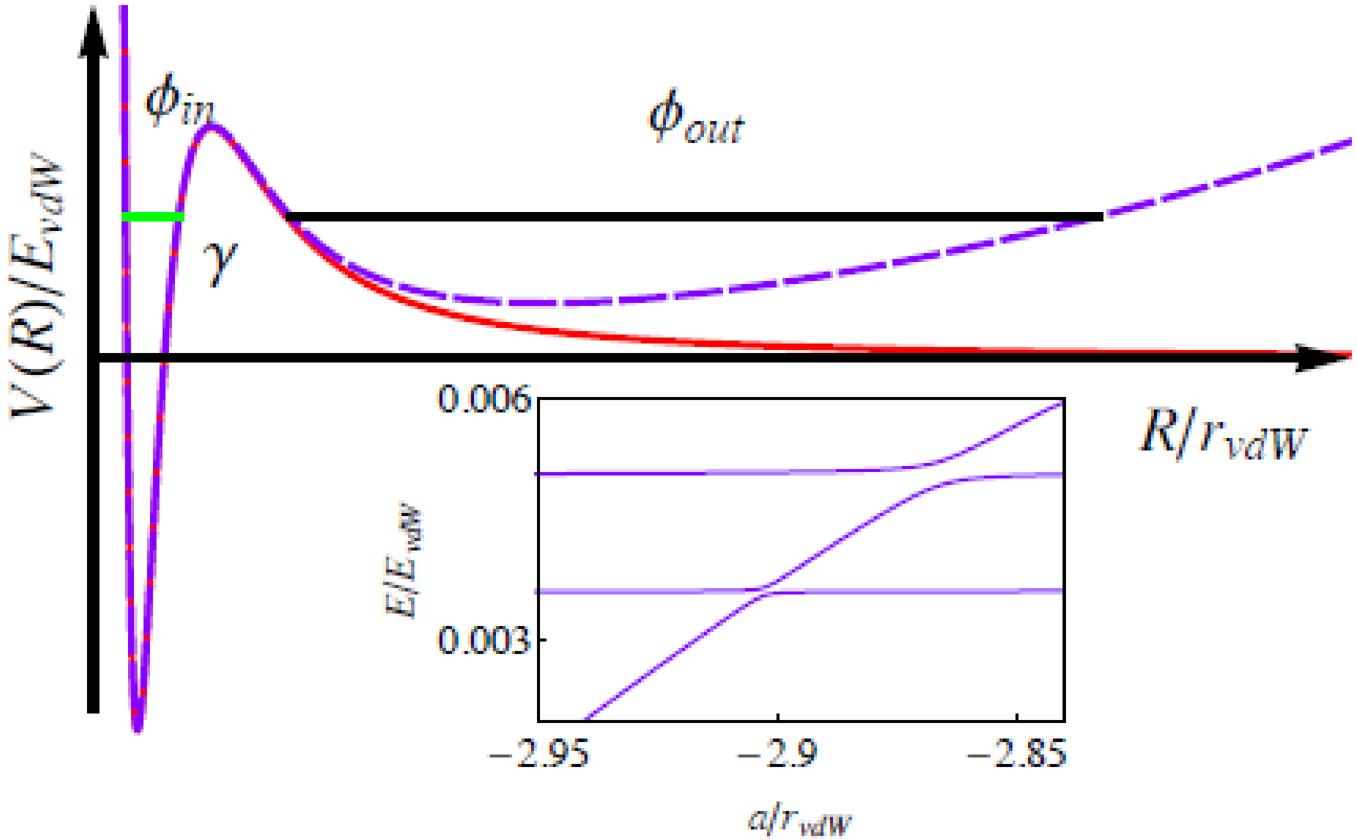
$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + \frac{p_4^2}{2m_4} + \frac{p_5^2}{2m_5}$$

$$+ V(r_{12}) + V(r_{13}) + V(r_{14}) + V(r_{15}) + V(r_{23}) + V(r_{24}) + V(r_{25}) + V(r_{34}) + V(r_{35}) + V(r_{45})$$

After eliminating the center-of-mass degree of freedom, we're left with a 12-dimensional PDE to solve, which can be reduced to **a mere 9 dimensions** for J=0 states after going to the body frame.

Resonant five-body recombination in an ultracold gas of bosonic atoms

Mulliken-style potential energy versus hyperradius R for 5 free Cs atoms (solid red) or harmonically trapped (



NJP 2013

Alessandro Zenesini[1]‡, Bo Huang[1], Martin Berninger[1], Stefan Besler[1], Hanns-Christoph Nägerl[1], Francesca Ferlaino[1], Rudolf Grimm[1,2], Chris H Greene[3]§, Javier von Stecher[3,4]

Our article with the Innsbruck group: finally published by the New Journal of Physics, accepted for publication about 1 year later!
Woohoo!

Resonant Five-Body Recombination in an Ultracold Gas

A. Zenesini, B. Huang, M. Berninger, S. Besler, H.-C. Nägerl, F. Ferlaino, and R. Grimm
Institut für Experimentalphysik and Zentrum für Quantenphysik, Universität Innsbruck, and
Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria

Chris H. Greene and J. von Stecher*
ity of Colorado, Boulder, CO 80309, USA
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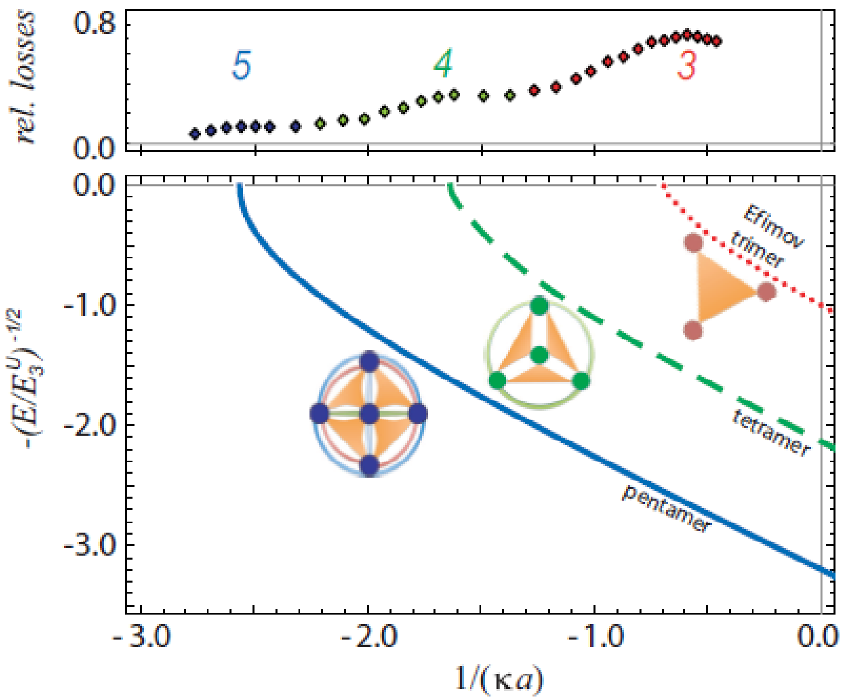
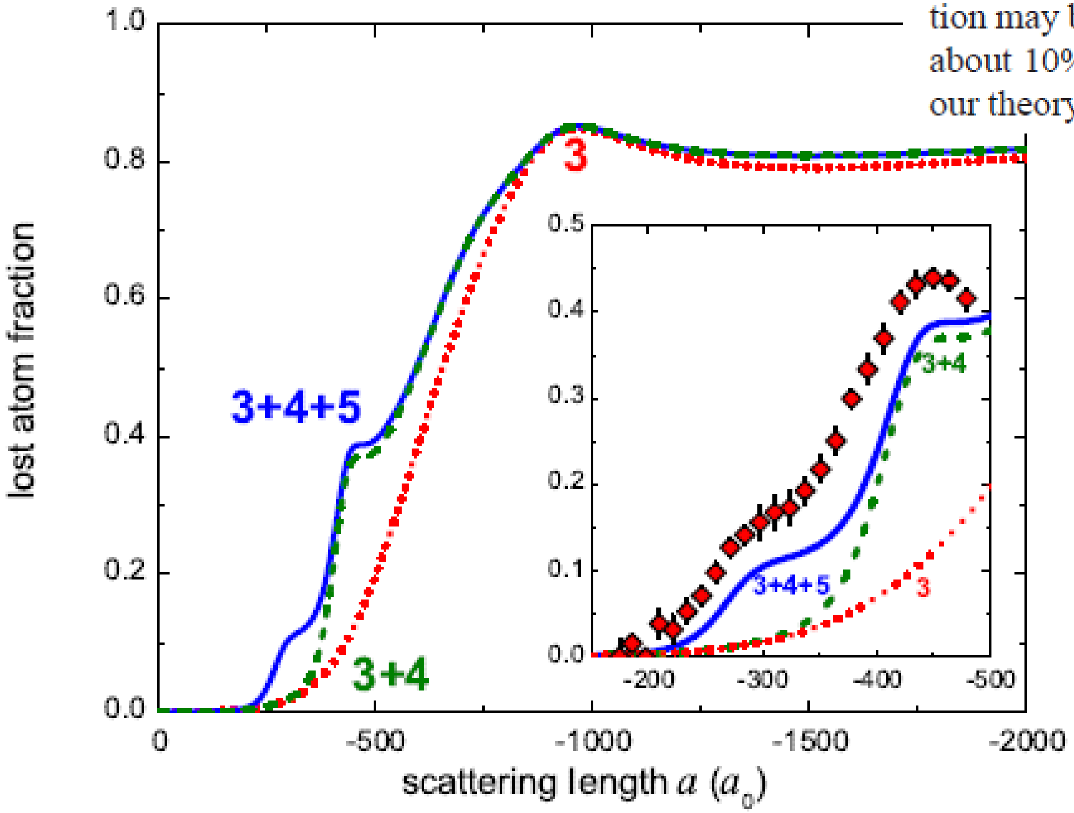


FIG. 1. (color online) N-body scenario in the region of negative two-body scattering length a. The lower panel shows the N-body binding energies as a function of the inverse scattering length. $E_3^U = (\hbar\kappa)^2/m$ is the trimer binding energy for resonant interaction. The dotted,

New Journal of Physics 15 (2013) 043040

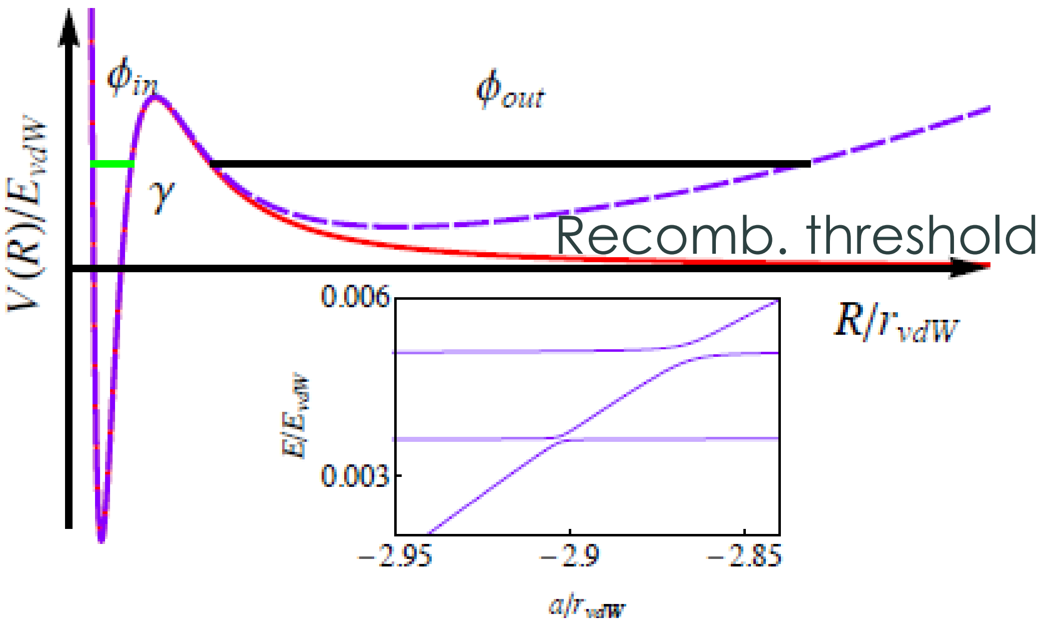
count for the experimental observations. Remarkably, the resonance position $a_{5,-} = 0.64(2)\,a_{4,-}$ is in agreement with the theoretical predictions $0.65(1)\,a_{4,-}$ [37, 38]. However, quantitatively, the experimental values for L_5 are about 15 times larger than the calculated ones. To account for this, we introduce a corresponding scaling factor. We find that this deviation may be explained by a small error in the WKB phase γ of about 10%, which remains in a realistic uncertainty range of our theory.



NJP 15 (2013) 043040, Zenesini et al.

FIG. 4. (color online) Calculated and measured fraction of loss atoms from an atomic sample of initially 5×10^4 atoms at a temperature of 80 nK after a hold time of 100 ms. The red dotted line corresponds to the losses predicted for three-body recombination only, while the dashed green line and the blue solid line include also contributions from four- and five body recombination, as quantified in this work. A

Schematic qualitative hyperspherical potential curve for 5 bosons at negative scattering length, “Mulliken style”



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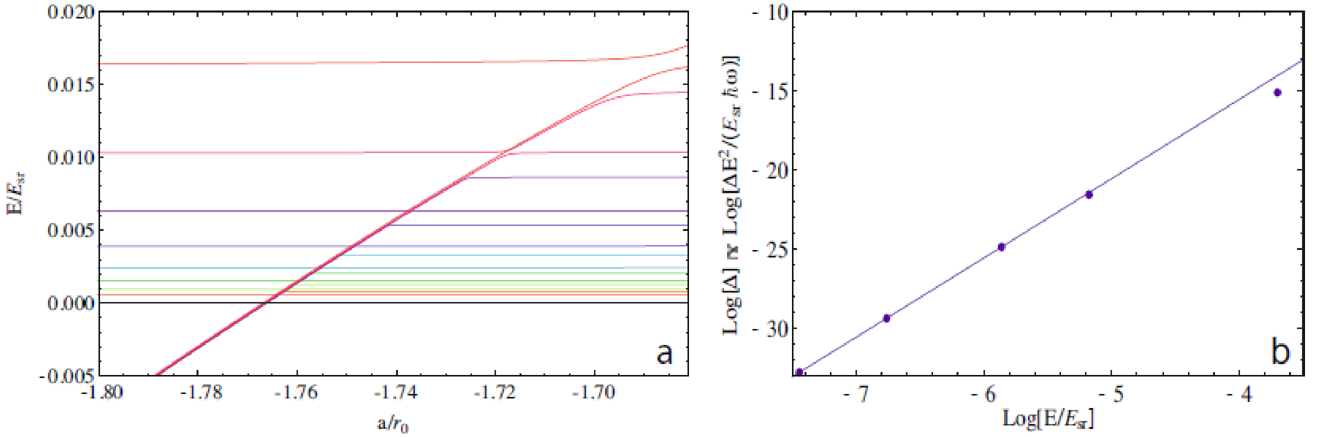


FIG. S1: (color online) (a) The lowest two eigenenergies of a trapped five body system are shown as functions of the scattering length for different trapping frequencies. Different colors represent different trapping frequencies. The combination of these states essentially describes the energy of the five-body state in the inner region of the potential $E_{mol}(a)$ (the diagonal curve). Here $E_{sr} = \hbar^2/(mr_0^2)$ and r_0 is the characteristic range of the two-body model potential that can be tuned to obtain the five-body resonance (i.e. $r_0 \sim 1.7 r_{vdw}$ where r_{vdw} is the van der Waals length). (b) The near-threshold behavior of Δ. The fitting of the lowest energy points leads implies that $\Delta \propto AE^b$. The lowest three points lead to $b \approx 5.004$ as expected from the known threshold behavior [4].

4-body recomb only

5-body recomb only

← **Separating the different N-body contributions**

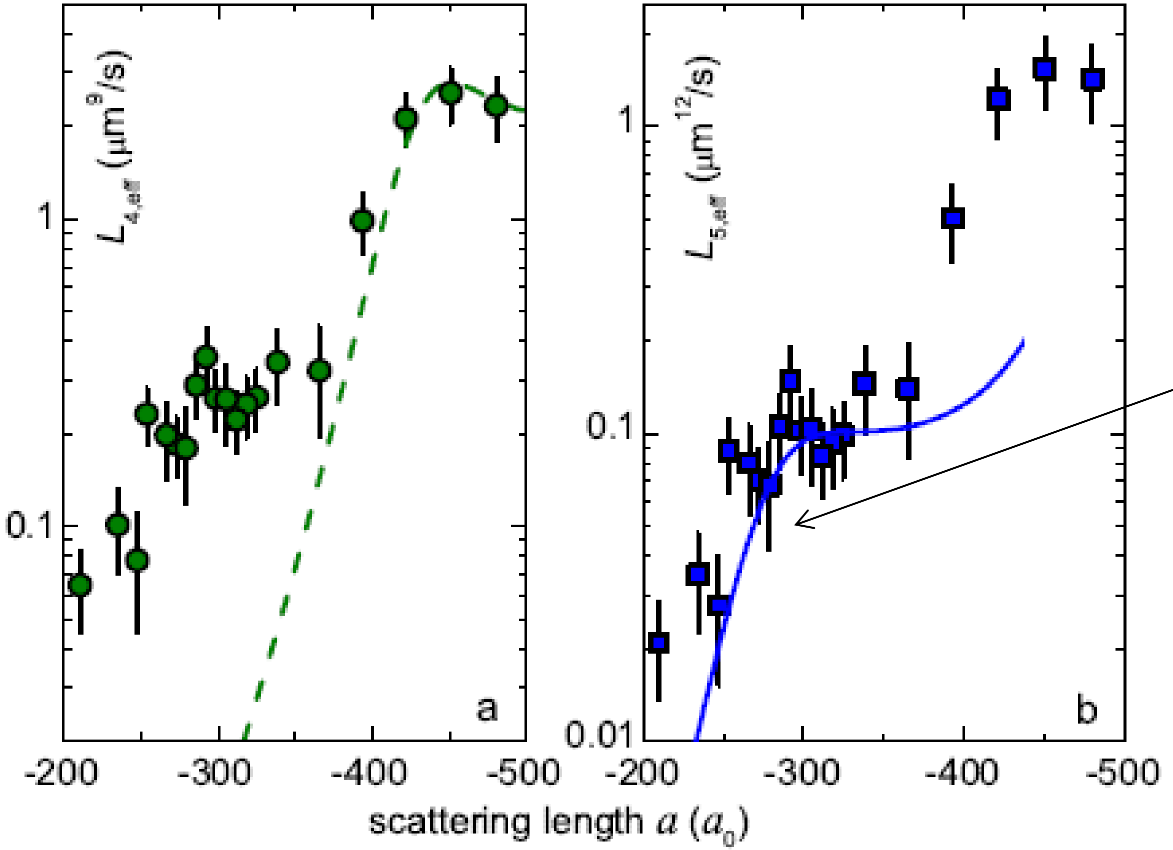


FIG. 3. (color online) Effective four- (a) and five-body recombination rates (b). The green dashed curve and the blue solid line follow the theoretical model for L_4 and L_5, respectively, with additional scaling factor for L_5; see text. The error bars include the statistical uncertainties from the fitting routine, the temperature and the trap frequencies.

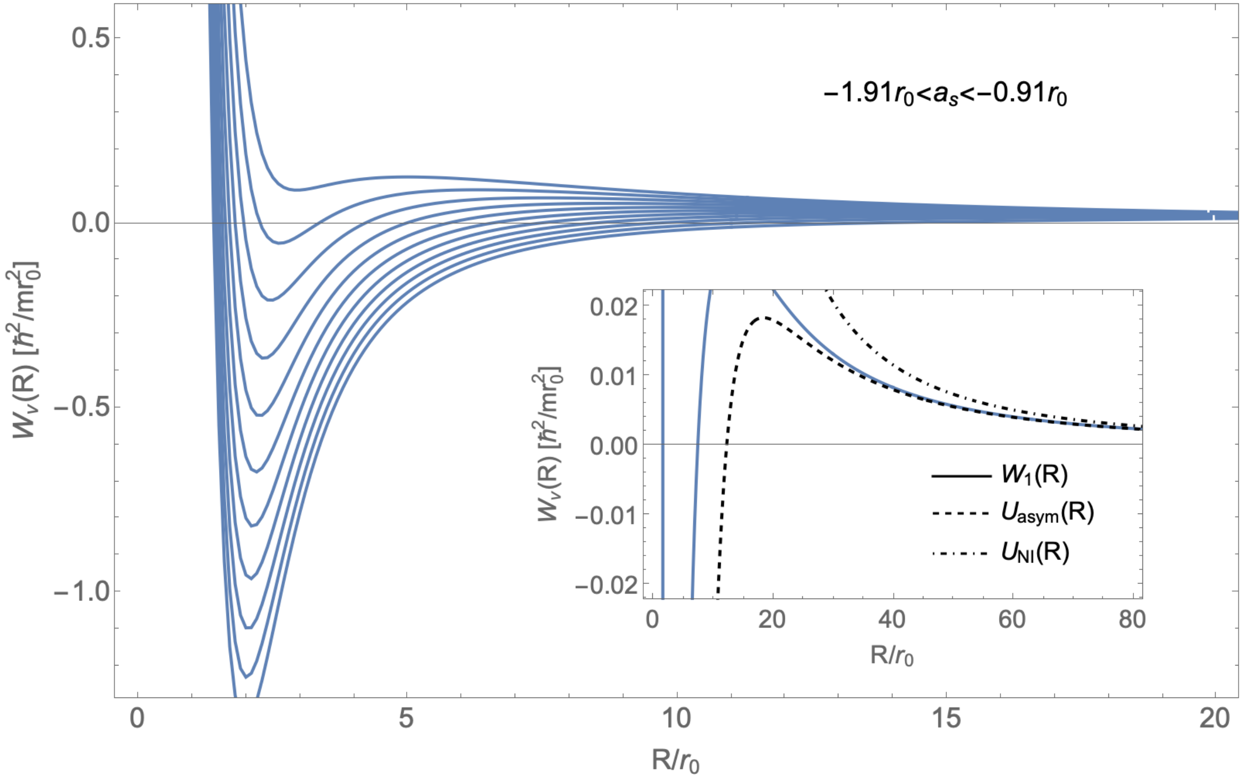
Position of the predicted 4-body resonance and the 5-body resonance is in agreement with experiment! Kewl!

NJP 15 (2013) 043040, Zenesini et al.

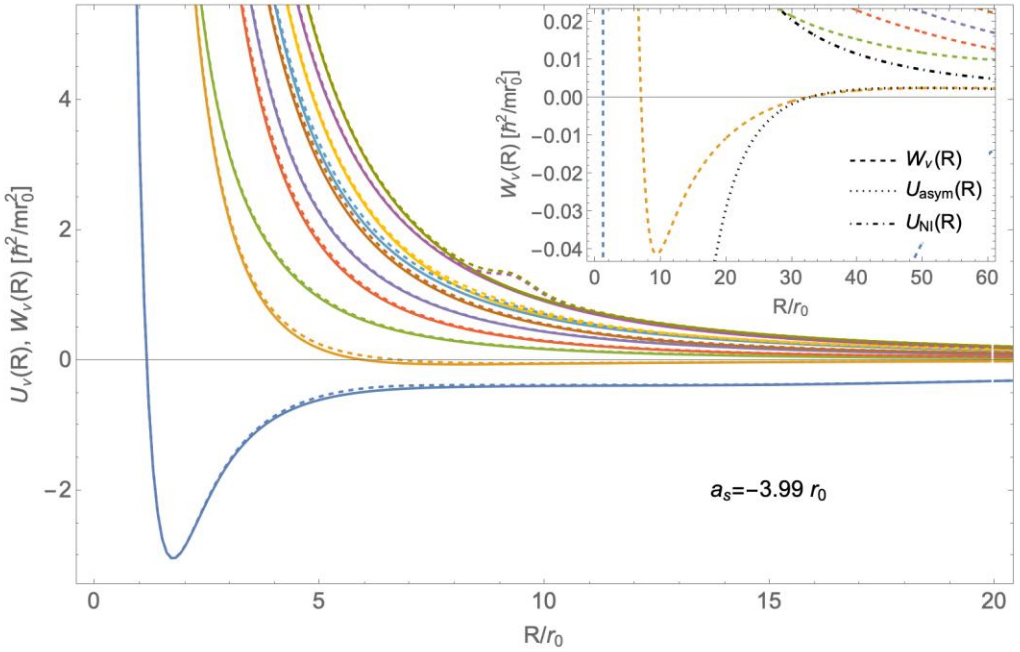
A combined theoretical and experimental study of 5-body recombination

At the time of our 5-body recombination article with the Innsbruck group, we could not yet calculate the actual 5-boson potential energy curve that serves as the entrance channel for 5-body recombination. Now, a talented (former) PhD student, now a postdoc in my group, Michael Higgins, has succeeded in computing it:

Lowest 5-boson potential curve vs. hyperradius, at several negative scattering lengths

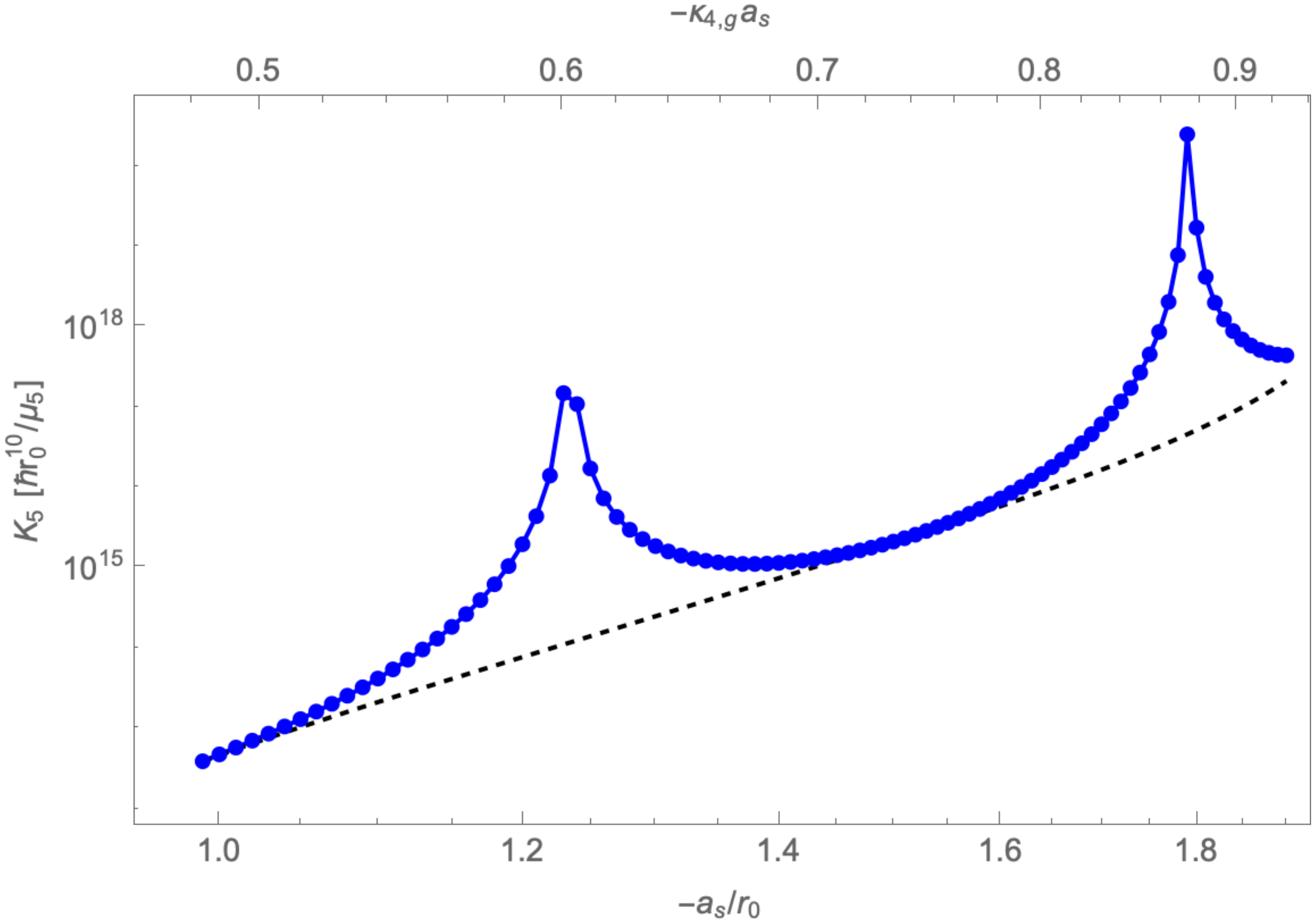


Lowest several potential curves at a more negative a where there is now one bound 4-boson universal state



PRELIMINARY!

Preliminary: calculated 5-boson recombination rate versus scattering length (note that there are TWO five-body resonances occurring before the first 4-body state binds



$$K_N^{0^+} = \frac{2\pi\hbar N!}{\mu\Omega(3N-3)}\left(\frac{2\pi}{k}\right)^{3N-5}\left(1-|S_{00}^{0^+}|^2\right)$$

Where in a WKB-style approximation the recombination probability is given by:

$$1-|S_{00}^{0^+}|^2 =$$

$$\frac{e^{-2\gamma}\sinh(2\eta)}{2(\cos^2(\phi)+\sinh^2(\eta))}\left|1+\frac{e^{-2\gamma}}{4}\tanh(\eta+\iota\phi)\right|^{-2}$$

Summary:

Our study of 3- and 4-fermion systems has given us a broader view of Efimov and faux-Efimov physics, with s-wave and/or p-wave interactions,

--plus-- we are learning how to think about threshold laws near unitarity

And, by the way, there are no 3n and no 4n bound or resonant states

We are gaining experience with 5-particle hyperspherical calculations, and our preliminary 5-boson recombination calculations look promising

One goal for the future:

adiabatic hyperspherical study of D+T → α+n

The adiabatic hyperspherical representation devised by Macek and promoted by the `Fano school' has promise for treating increasingly complex systems, but difficulties remain. Nevertheless,……"One must try" (Dirac)