Towards the Unitarity Limit in EFTs with Pions

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1. Emergent Phenomena in Nuclear Physics: “Order From Chaos”
2. What Is The Unitarity Limit? And Why Should I Care?
3. Unitarity Expansion With Perturbative Pions in NN
4. Concluding Hypothesis and Questions

How to root Nuclear Physics in QCD?
What is the underlying principle that makes simple structures emerge from complex nuclear dynamics?

König/hg/Hammer/van Kolck: *Phys. Rev. Lett.* **118** (2017) 202501 [1607.04623 [nucl-th]]

Teng/hg: MSc Thesis GW 2023 and [2410.09653 [nucl-th]]

1. Emergent Phenomena in Nuclear Physics: “Order From Chaos”

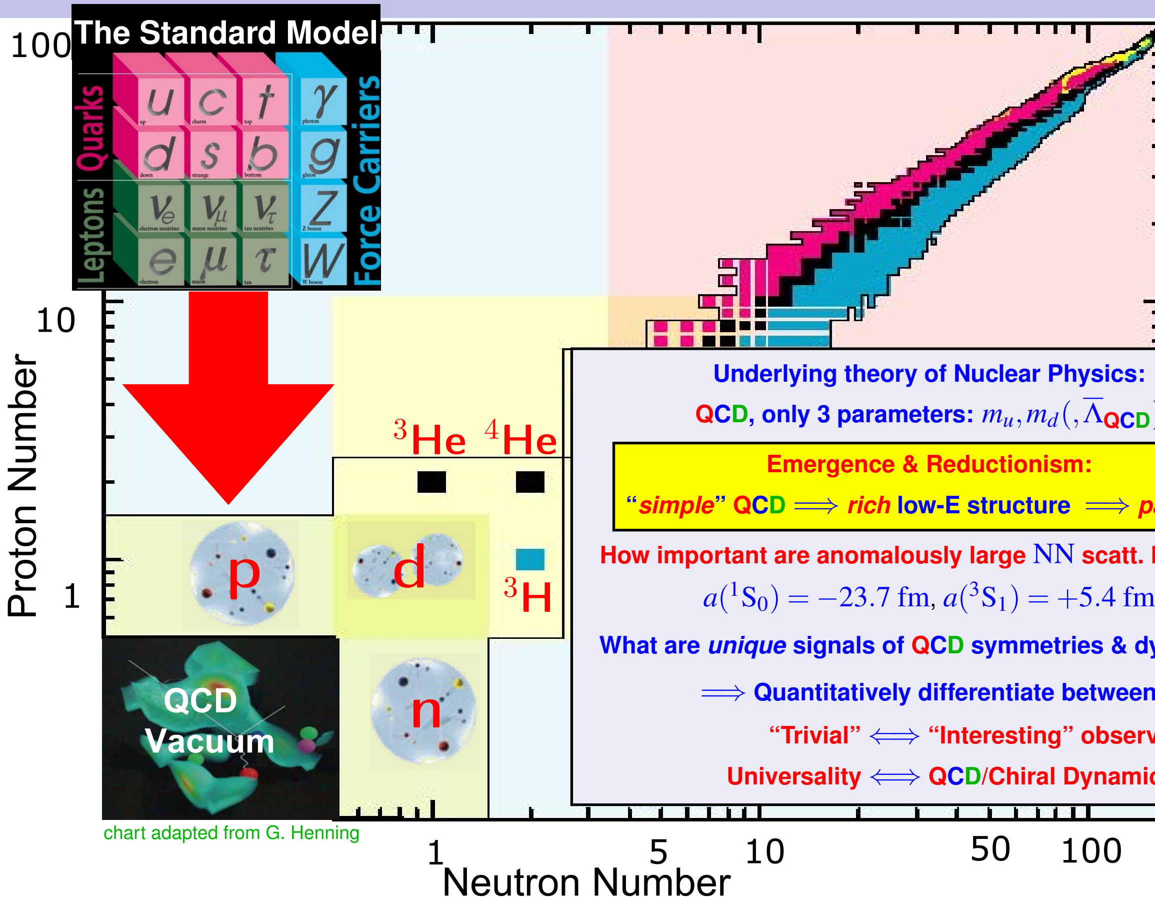


chart adapted from G. Henning

Underlying theory of Nuclear Physics:

QCD, only 3 parameters: $m_u, m_d (, \overline{\Lambda}_{\mathrm{QCD}})$

Emergence & Reductionism:

“*simple*” QCD $\Longrightarrow$ *rich* low-E structure $\Longrightarrow$ *patterns*

How important are anomalously large NN scatt. lengths?:

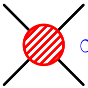
$a(^1\mathrm{S}_0) = -23.7$ fm, $a(^3\mathrm{S}_1) = +5.4$ fm

What are *unique* signals of QCD symmetries & dynamics?

$\Longrightarrow$ Quantitatively differentiate between:

“Trivial” $\Longleftrightarrow$ “Interesting” observables

Universality $\Longleftrightarrow$ QCD/Chiral Dynamics

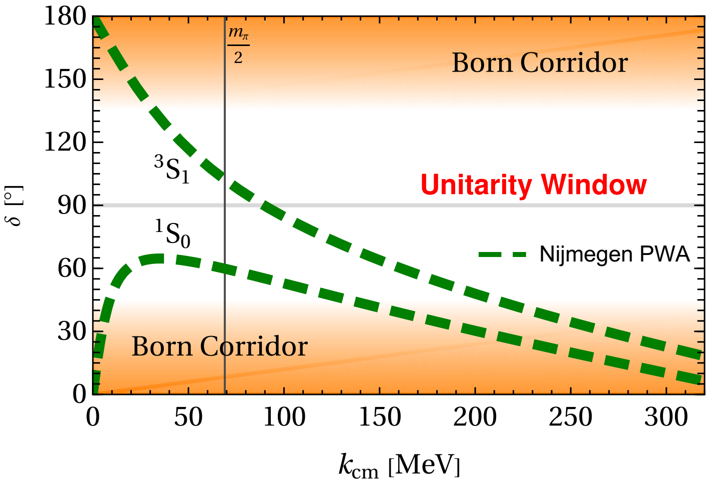


$$\propto \frac{1}{\underbrace{k\cot\delta}_{\text{interaction}} - \mathrm{i}k} \to \begin{cases} \dfrac{1}{k\cot\delta}\left[1+\dfrac{\mathrm{i}}{\cot\delta}+\dots\right] & \text{for } \cot\delta \gg |\mathrm{i}| \text{ i.e. } \delta\to 0 \implies \\ \dfrac{1}{-\mathrm{i}k}\left[1+\dfrac{\cot\delta}{\mathrm{i}}+\dots\right] & \text{for } \cot\delta \ll |\mathrm{i}| \text{ i.e. } \delta\to 90^\circ \implies \end{cases}$$

interaction ↓ Unitarity

Born Approximation:
interactions small & perturbative,
their details & scales drive A_{NN}
no bound states

Unitarity Limit implies Universality:
interaction strong: *non-perturbative*,
details irrelevant, unitarity drives A_{NN};
Unitarity Expansion at LO:
no scales in A_{NN}, bound state at $k=0$.



Unitarity Window: $|\cot\delta| \lesssim 1$ ($45^\circ \lesssim \delta \lesssim 135^\circ$)

$\implies$ LO NN nonperturbative in $^1S_0$ & $^3S_1$ for

$30\,\mathrm{MeV} \lesssim k_{\mathrm{cm}} \lesssim [1.5\dots2]m_\pi$

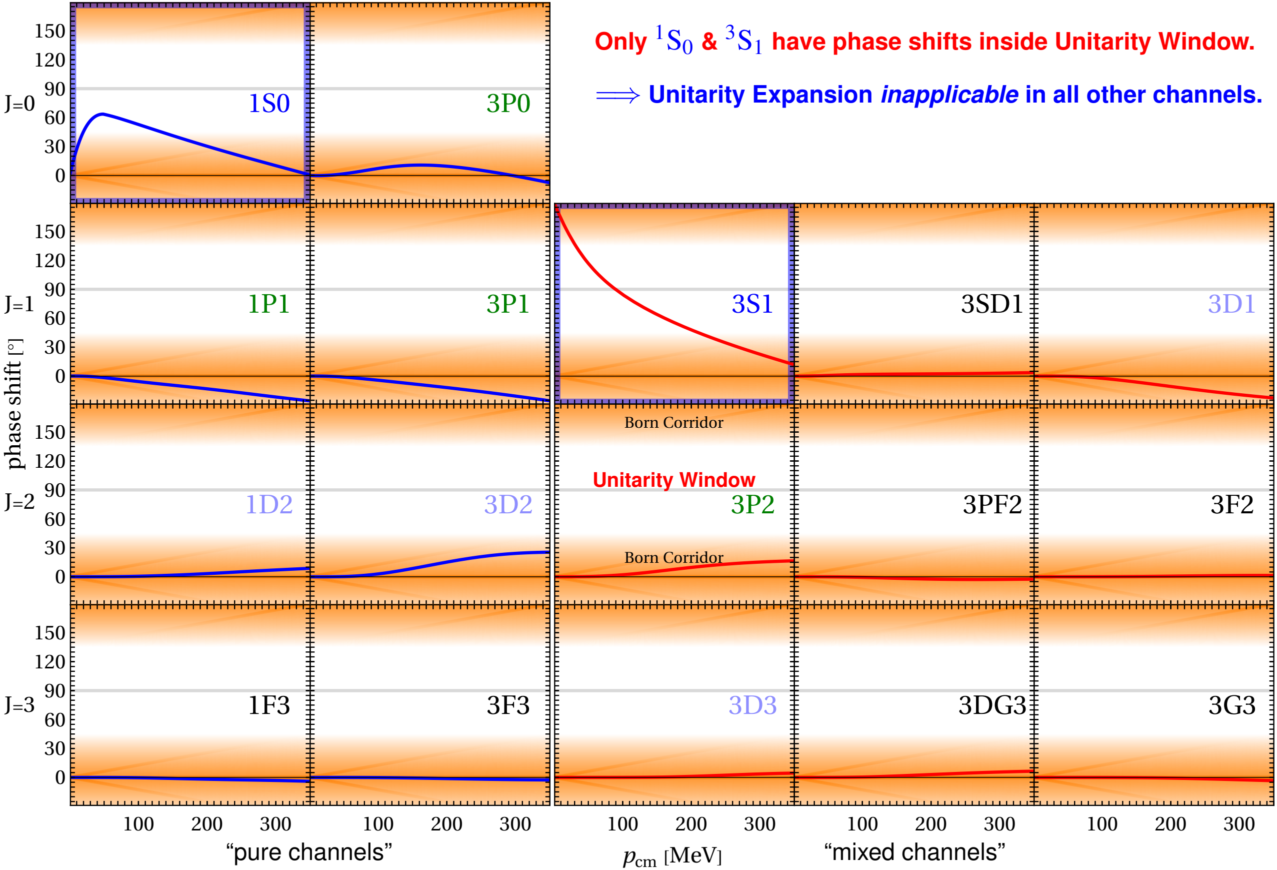
Outside: **Born Corridors**

LO perturbative for $|\cot\delta| \gtrsim 1$ ($|\delta| \lesssim 45^\circ$)

How much of Nuclear Physics does really depend on details of QCD?

How much just from (corrections to) universal aspects around Unitarity?

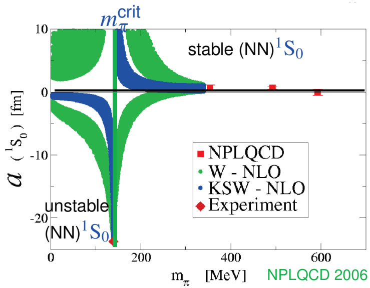
(a) Use Unitarity Expansion Only for Channels with Large NN Phase Shifts!



Only $^1S_0$ & $^3S_1$ have phase shifts inside Unitarity Window.

$\Longrightarrow$ Unitarity Expansion *inapplicable* in all other channels.

(b) Symmetries in the Unitarity Limit



χEFT cannot explain anomalous scatt. lengths/shallow binding: Worlds with $a \lesssim \frac{1}{m_\pi}$!

Noether Theorem 1918 [physics/0503066]:

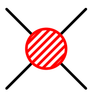
Symmetries and their breaking result in conserved quantities.

$$k\cot\delta = 0 \Longleftrightarrow S = e^{2i(\delta=\frac{\pi}{2})} = -\mathbb{1}$$

(1) **Amplitude saturated at Unitarity Limit:** $\sigma = \frac{4\pi}{k^2}$ maximal (probability conservation).

(2) **Scale Invariance:** $\vec{k} \to e^\lambda\,\vec{k}$. and Conformal Symmetry...

(3) **Wigner-$SU(4)$ Symmetry of combined spin-isospin rotations** $\begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix} \to U \begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix}$ Wigner, Hund 1937 for heavy nuclei cf. Mehen/Stewart/Wise 1999

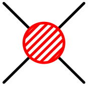
In NN:  $= \frac{4\pi}{M}\,\frac{1}{-\frac{1}{a} - ik} = A_{NN}(^3S_1) = A_{NN}(^1S_0)$ if $a(^3S_1) = a(^1S_0)$.

Theorists love Unitarity Limit as Nontrivial Fixed Point characterised by high symmetry:
Wigner-$SU(4)+$ scale-invariance close to FP protected in renormalisation.

What About Nature?

(c) Unitarity Expansion in EFT($\not\pi$)

König/hg/Hammer/van Kolck: PRL 2017 [1607.04623]
reviews: van Kolck [2003.09974]; Kievsky/... [2102.13504]

EFT($\not\pi$)/ERE:  $\propto \frac{1}{-ik}\left[1+\frac{k\cot\delta = -\frac{1}{a}+\frac{r}{2}k^2+\dots}{ik}+\dots\right] \to \underbrace{\frac{1}{-ik}}_{\text{LO}}\left[1+\mathrm{i}\underbrace{\left(\underbrace{\frac{1}{ka}}_{<1?!}-\underbrace{\frac{kr}{2}}_{<1?!}\right)}_{\text{NLO correction}}+\dots\right]$

***a priori* justified if** inverse scatt. length/ NN system size/ NN binding momentum $0 \leftarrow \frac{1}{a} \ll$ **typ. momentum** $k \ll \Lambda_{\not\pi} \overset{???}{\sim} m_\pi \sim \frac{1}{r}$ **breakdown/ resolution scale.**

LO: No NN scale. $\Longrightarrow$ Nuclear Physics correlated to *just one* 3N *RG scale fixed by* B_3 *via Efimov effect*.

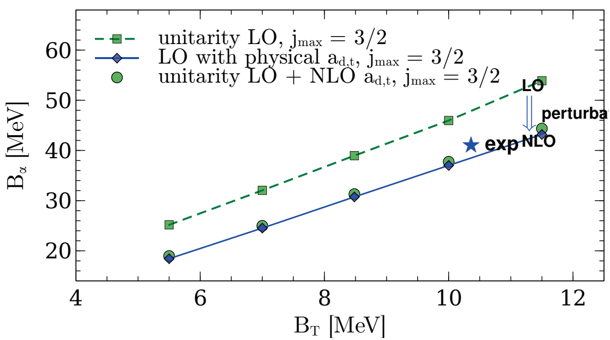
PARADIGM SHIFT: *Unitarity* de-emphasises details of NN & pions, emphasises 3N scale & Universality.

Information Theory in EFT: lossless compression into smallest number of parameters at given accuracy.

$\Longrightarrow$ Explore *Sweet Spot* for patterns, unique signals of QCD:

bound weakly enough to be insensitive to interaction details ($\frac{kr}{2} \ll 1$),

but strongly enough to be insensitive to exact large system size ($ka \gg 1$).

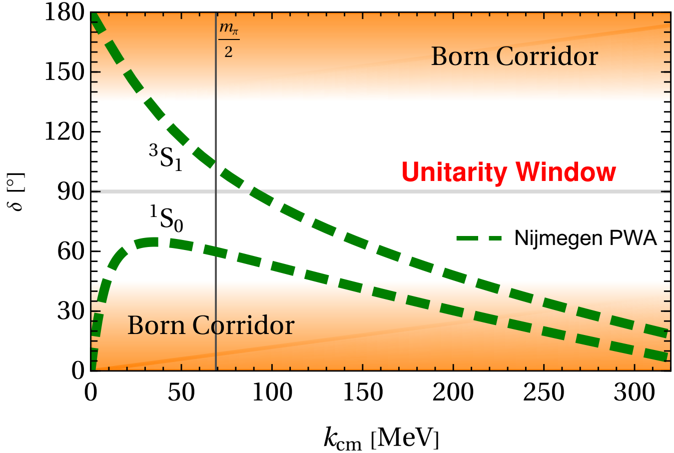


$B_{^3\mathrm{H}} - B_{^3\mathrm{He}}$: NLO: $[0.92 \pm 0.18]$ MeV; exp: 0.764 MeV

	Fermion Unitarity LO $\to$ NLO	exp ^{4}He/^{3}H
ground: B_4/B_3	$4.6 \to 3.8 \pm 0.2$	3.66
excited: B_4^*/B_3	$\sim 1.1 \to \sim 0.98 \pm 0.05$	0.96

Symm. Nucl. Matter	ρ_0 [fm^{-3}]	B/A [MeV]	E_{sym} [MeV]	L[MeV] slope of E_{sym}	K_∞[MeV] compressib.
Kievsky/... EFT($\not\pi$)-inspired	0.15	-16	35	70	251
exp	0.16	-16	≈ 30	$[40\dots60]$	210

(d) χEFT Should Work In the Unitarity Expansion!



NN S waves well in **Unitarity Window** $|\cot\delta| < 1$

for $30\text{ MeV} \lesssim k_{cm} \lesssim [1.5\ldots 2] m_\pi$.

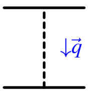
Window's upper limit close to scale

$$\overline{\Lambda}_{NN} = \frac{16\pi f_\pi^2}{g_A^2 M} \approx 300\ \text{MeV}$$

where OPE becomes nonperturbative KSW 1999, FMS 2000.

$\Longrightarrow$ **How to embed pions/χiral symmetry inside Unitarity Window?**

Problem: Pions break **scaling by f_π, m_π, Wigner by mixing.**



$$: \ -\frac{g_A^2}{4 f_\pi^2}\,\frac{1}{\vec{q}^2 + m_\pi^2}\left[\underbrace{\frac{1}{3}\,(\vec{\tau}_1\cdot\vec{\tau}_2)\,(\vec{\sigma}_1\cdot\vec{\sigma}_2)\,\vec{q}^2}_{\text{central: Wigner-Irrep}} + \underbrace{(\vec{\tau}_1\cdot\vec{\tau}_2)\left((\vec{\sigma}_1\cdot\vec{q})(\vec{\sigma}_2\cdot\vec{q}) - \frac{1}{3}\,(\vec{\sigma}_1\cdot\vec{\sigma}_2)\,\vec{q}^2\right)}_{\text{tensor: Wigner-breaking, mixes } S\leftrightarrow D,\ D\to D}\right]$$

Explore transition "no$\longrightarrow$nonperturbative pions" via Perturbative ("KSW") Pions (only undisputedly consistent χEFT).

χEFT(pπ)$_{UE}$: χEFT with Perturbative Pions in the Unitarity Expansion: $Q \sim \frac{1}{ka}, \frac{kr}{2}, \frac{k, m_\pi}{\overline{\Lambda}_{NN}} \ll 1$

$\Longrightarrow$ Apply Unitarity Expansion to N^{2}LO amplitudes already computed analytically

by Rupak/Shoresh PRC60 (2000) 0540004 [nucl-th/9902077] ($^1S_0$) and Fleming/Mehen/Stewart NPA677 (2000) 313 [nucl-th/9911001] ($^1S_0$, $^3S_1$).

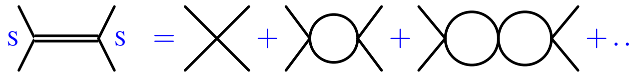
3. Unitarity Expansion With Perturbative Pions in NN

(a) χEFT(pπ)$_{\text{UE}}$ at N^{2}LO with $Q \sim \frac{1}{ka}, \frac{kr}{2}, \frac{k, m_\pi}{\Lambda_{\text{NN}}} \ll 1$

based on Rupak/Shoresh [nucl-th/9902077] ($^1\mathrm{S}_0$), Fleming/Mehen/Stewart [nucl-th/9911001] ($^1\mathrm{S}_0$, $^3\mathrm{S}_1$)
mod. for unitarity Teng/hg MSc thesis GW 2023, [2410.09653]

$\mathcal{O}(Q^{-1})$ **(LO):** Nonperturbative; no scale, perfect Wigner, pure S wave.

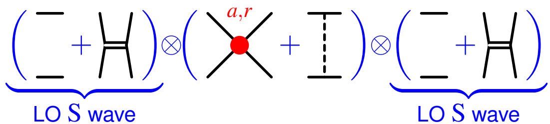
$$A^{(\mathrm{S})}_{-1} = \frac{4\pi \mathrm{i}}{M}\,\frac{1}{k} =$$



$\mathcal{O}(Q^{0})$ **(NLO):** Scaling and Wigner broken by contacts determined to reproduce PWA values of a, r.

Non-iterated OPE: central only, does not break Wigner but scaling; first non-analyticity: branch point $\pm \mathrm{i}\frac{m_\pi}{2}$.

$$A^{(\mathrm{S})}_{0} =$$



$\Longrightarrow$ **Unitarity, Wigner-$\mathrm{SU}(4)$ spin-isospin symmetry align naturally for Perturbative Pions at NLO.**

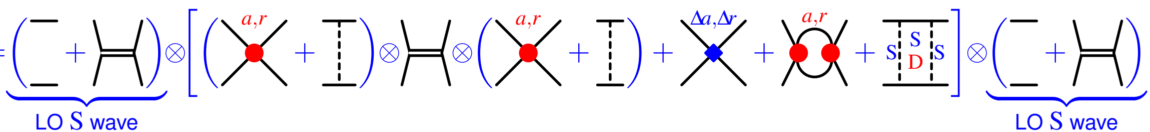
$\mathcal{O}(Q^{1})$ **(N^{2}LO):** Contacts adjusted to keep a, r at PWA values; multiplied with non-iterated OPE (central only).

Once-iterated OPE added: first and second non-analyticity: branch points $\pm \mathrm{i}\frac{m_\pi}{2}, \pm \mathrm{i} m_\pi$.

$A_{1\text{sym}}$: Central $\mathrm{S} \to \mathrm{S} \to \mathrm{S}$ does not break Wigner but scaling: identical in $^1\mathrm{S}_0$ and $^3\mathrm{S}_1$.

$A_{1\text{break}}$: Tensor $\mathrm{S} \to \mathrm{D} \to \mathrm{S}$ breaks Wigner and scaling: only in $^3\mathrm{S}_1$.

$$A^{(\mathrm{S})}_{1} =$$



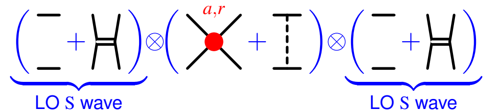
$\Longrightarrow$ **Is breaking of Wigner-$\mathrm{SU}(4)$ spin-isospin symmetry for Perturbative Pions at N^{2}LO indeed small?**

(b) Analytic Answers Shorter By Unitarity

based on Rupak/Shoresh [nucl-th/9902077] ($^1S_0$),
Fleming/Mehen/Stewart [nucl-th/9911001] ($^1S_0$, $^3S_1$)
mod. for unitarity Teng/hg MSc thesis GW 2023, [2410.09653]

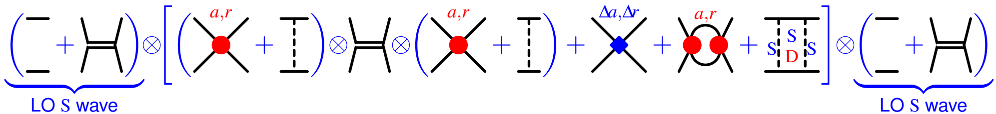
LO: $A^{(S)}_{-1}(k) = \frac{4\pi i}{M}\,\frac{1}{k}$ is only S wave

NLO: $$A^{(S)}_0(k) = -\frac{4\pi}{Mk}\left(\frac{1}{ka}-\frac{kr}{2}\right) - \frac{g_A^2}{4f_\pi^2}\left(1-\frac{m_\pi^2}{4k^2}\overbrace{\ln[1+\frac{4k^2}{m_\pi^2}]}^{1\pi\text{ cut}}\right)$$



Non-iterated OPE does not break Wigner.

N^{2}LO:



Once-iterated OPE breaks Wigner: $S \to D \to S$

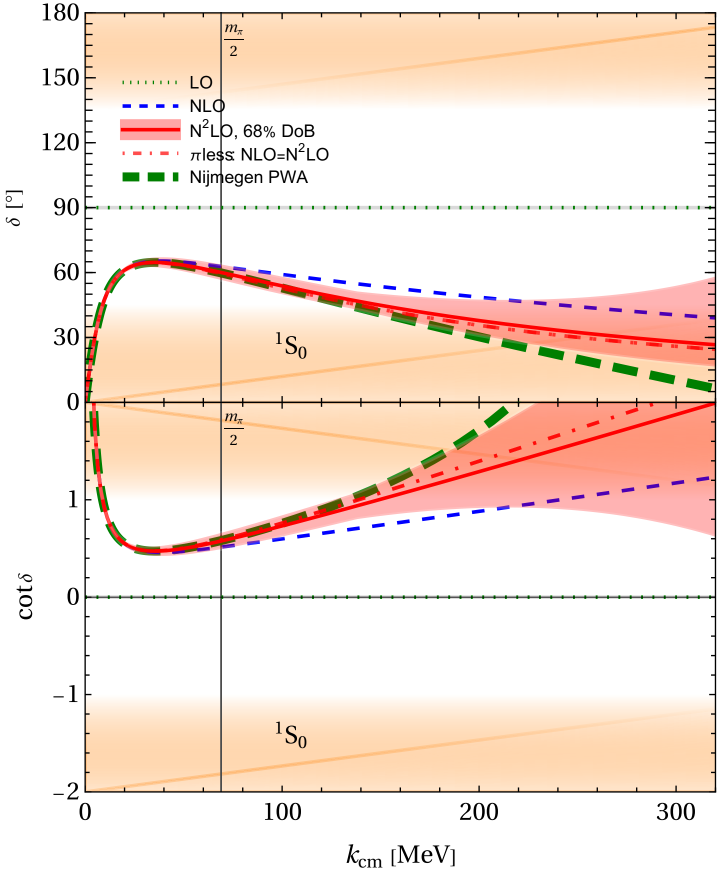
$$A_1^{(^1S_0)}(k) \equiv A^{(S)}_{1\text{sym}}(k) = \frac{\left[A_0^{(S)}(k)\right]^2}{A^{(S)}_{-1}(k)} + \frac{g_A^2}{2f_\pi^2}\left[\frac{4}{3am_\pi} - \frac{m_\pi}{k}\left(\frac{1}{ka}-\frac{kr}{2}\right)\right]$$
$$- \frac{g_A^2 M m_\pi}{16\pi f_\pi^2}\left\{A_0^{(S)}(k)\,\frac{m_\pi}{k}\underbrace{\arctan[\frac{2k}{m_\pi}]}_{1\pi\text{ cut}} + \frac{g_A^2}{2f_\pi^2}\left[\frac{1}{12}+\left(\frac{m_\pi^2}{4k^2}-\frac{1}{3}\right)\ln 2 - \underbrace{F_\pi(\frac{k}{m_\pi})}_{1,2\pi\text{ cut}}\right]\right\}$$

$$A_1^{(^3S_1)}(k) = A^{(S)}_{1\text{sym}}(k) + A^{(S)}_{1\text{break}}(k)$$

$$A^{(S)}_{1\text{break}}(k) = -\frac{\left[A_0^{(SD)}(k)\right]^2}{A^{(S)}_{-1}} + \frac{g_A^2}{f_\pi^2}\frac{g_A^2 M m_\pi}{16\pi f_\pi^2}\left\{\frac{571-352\ln 2}{210} - \left(1+\frac{3m_\pi^2}{2k^2}+\frac{9m_\pi^4}{16k^4}\right)\overbrace{F_\pi(\frac{k}{m_\pi})}^{1,2\pi\text{ cut}} + \frac{2m_\pi^2}{5k^2}(\ln 4 - 1)\right.$$
$$\left. + \frac{3m_\pi^4}{16k^4} - \frac{3}{2}\left[\frac{k}{m_\pi}+\frac{m_\pi}{k}-\frac{m_\pi^3}{8k^3}-\frac{3m_\pi^5}{16k^5}\right]\overbrace{\arctan[\frac{k}{m_\pi}]}^{2\pi\text{ cut}} + \frac{3}{16}\left(\frac{m_\pi^4}{k^4}+\frac{3m_\pi^6}{4k^6}\right)\overbrace{\ln[\frac{16(k^2+m_\pi^2)}{4k^2+m_\pi^2}]}^{1,2\pi\text{ cut}}\right\}$$

$$F_\pi(x) := \frac{1}{8x^3}\left(\underbrace{\arctan[2x]\ln[1+4x^2] - \mathrm{Im}\left[\mathrm{Li}_2[\frac{2ix+1}{2ix-1}]\right.}_{1\pi\text{ cut}} \left. - 2\,\underbrace{\mathrm{Li}_2[\frac{1}{2ix-1}]}_{2\pi\text{ cut}}\right]\right)$$

(c) Perturbative Pions at N^{2}LO: $^1S_0$



$^1S_0$: central OPE $\Longrightarrow$ Wigner-symmetric.

f_π, m_π break scaling.

Strict perturbation in "basic interaction part"

$$k\cot\delta = 0_{\rm LO} + k\cot\delta_{\rm NLO} + k\cot\delta_{\rm N^2LO}$$
$$= 0_{\rm LO} - \frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2}\right].$$

$\Longrightarrow$ Get δ from $k\cot\delta$.

$^1S_0$ is **"boring"** partial wave: no tensor int.

Bayesian truncation uncertainty at 68% DoB.

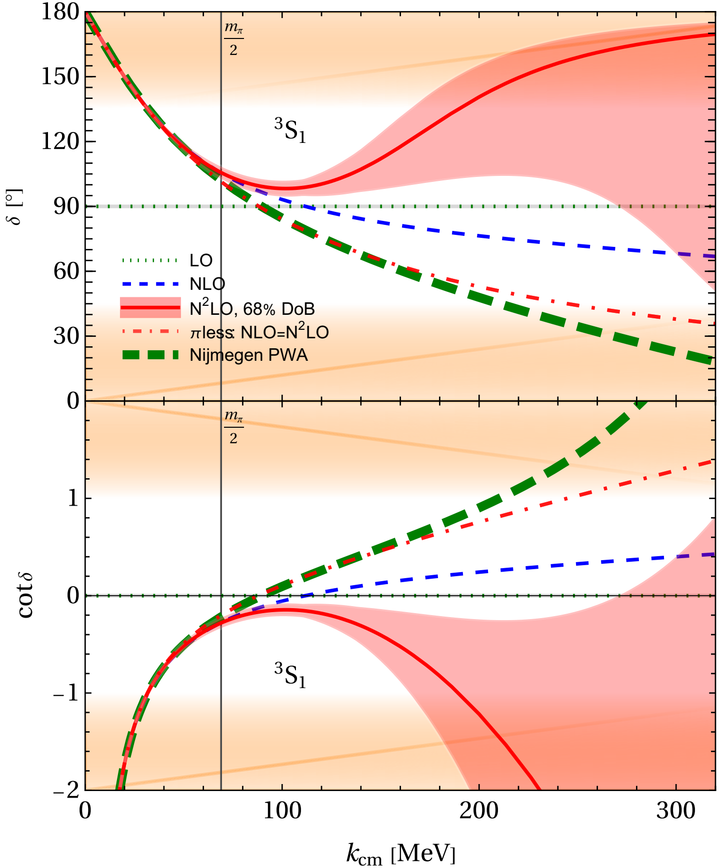
$\Longrightarrow$ Converges order-by-order $\lesssim 300$ MeV.

Agrees within uncertainties with PWA for $\lesssim 250$MeV (even outside Unitarity Window).

Compare to EFT($\not\pi$): minuscule impact of π.

(d) Perturbative Pions at N^{2}LO: $^3S_1$

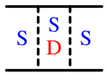
perturbative pions to N^{2}LO: Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



$^3S_1$: pions break Wigner-$SU(4)$ & scale inv.

$^3S_1$ is **"interesting"** partial wave:

tensor-OPE $\Longrightarrow$ SD mixing from



$$k\cot\delta = -\frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2} + \frac{A_{SD}^2}{A_{-1}^3}\right]$$

$\Longrightarrow$ Terrible convergence (already in FMS):

Converges order-by-order $\lesssim 80$ MeV.

Agreees within uncertainties with PWA only for $\lesssim 70$ MeV (not even in Unitarity Window).

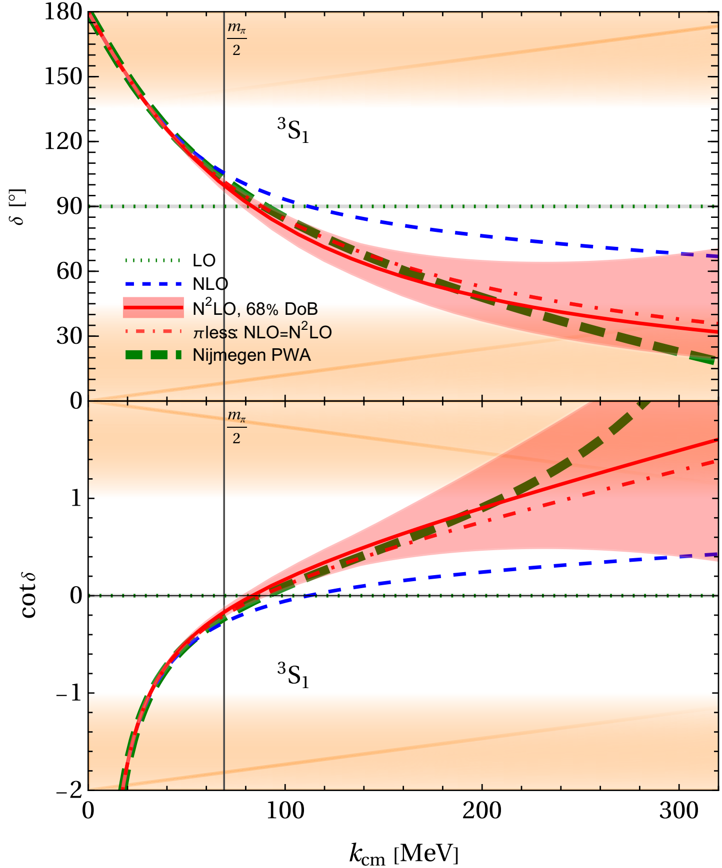
Compare to EFT($\not\pi$): huge impact of pion.

Source of discrepancy: tensor int. (via SD).

(central is identical in $^3S_1$&$^1S_0$)

(d) Perturbative Pions at N^{2}LO: $^3S_1$

perturbative pions to N^{2}LO: Fleming/Mehen/Stewart 2000
unitarity for them: Teng/hg MSc Thesis GW 2023, [2410.09653]



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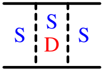


Broken Wigner-$SU(4)$ spoils convergence!

Idea: Use Wigner-$SU(4)$-symmetric pion part.

$\Longrightarrow$ Only $^1S_0$-$^3S_1$ differences of a & r break Wigner-$SU(4)$.

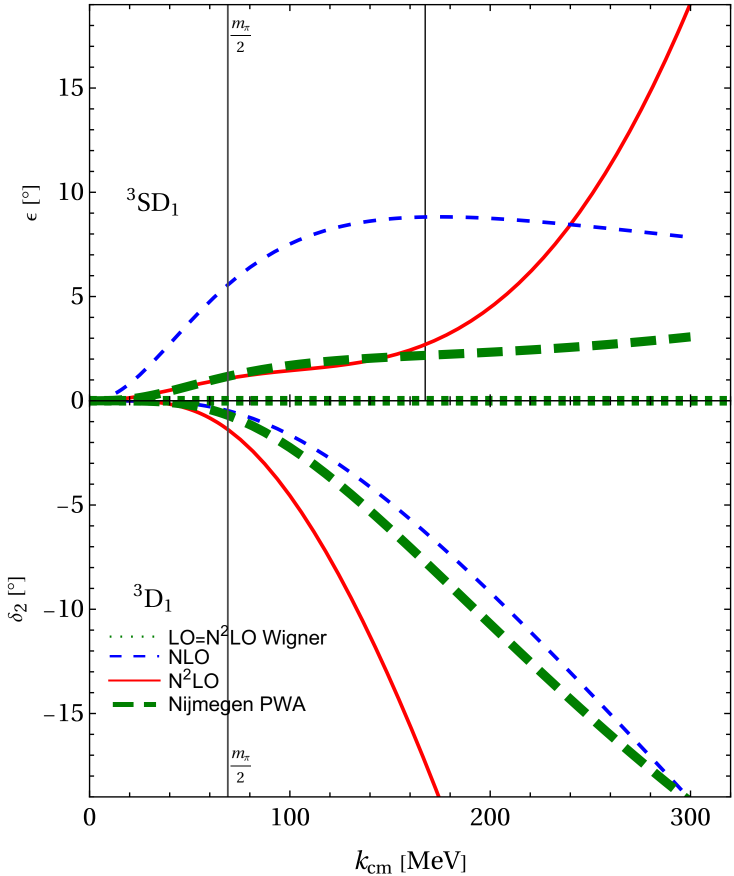
RG-invariant, mildly χsymmetry-breaking.

$\Longrightarrow$ Converges order-by-order $\gtrsim 300$ MeV.

Agrees within uncertainties with PWA for $\gtrsim 300$ MeV (even outside Unitarity Window).

Compare to EFT($\not\pi$): tiny impact of pion.

$\Longrightarrow$ **All very similar to $^1S_0$.**



No other channels close to Unitarity Window:

$$|\delta_{l\geq 1}| < 25^\circ \; (|\cot\delta_{l\geq 1} > 2|).$$

$^3SD_1$ mixing only by tensor/Wigner-breaking.

In Unitarity Expansion very similar to FMS:

$k \gtrsim 70$ MeV:

No order-by-order convergence,

convergence to PWA elusive.

Zero by Wigner at N^{2}LO.

Natural size at N^{3}LO at $k \approx m_\pi$:

$90^\circ \times (Q \approx 0.5)^3 \approx 10^\circ$

$\Longleftrightarrow$ PWA: $\lesssim 10^\circ$.

$\Longrightarrow$ Not inconsistent.

SD & DD contacts at N^{3}LO

$\Longrightarrow$ Reproducing PWA possible.

4. Concluding Hypothesis and Questions

Teng/hg MSc Thesis GW 2023, [2410.09653]

χEFT with Perturbative Pions in Unitarity Expansion $Q \sim \frac{1}{ka}, \frac{kr}{2}, \frac{m_\pi}{\overline{\Lambda}_{\rm NN}} \ll 1$: needs $\delta \to \frac{\pi}{2} \Longrightarrow {}^1\mathrm{S}_0, {}^3\mathrm{S}_1$ only!

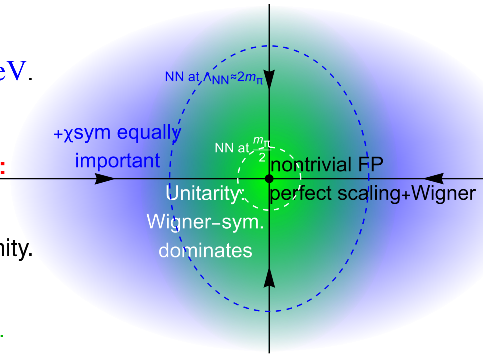
Chiral Physics: $m_\pi, f_\pi, (\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})(\tau_1 \cdot \tau_2)$ seem opposed to Wigner, but NN/few-N projection forces into it.

Hypothesis (at least for Perturbative Pions): Tensor/Wigner-$\mathrm{SU}(4)$ symmetry-breaking part of One-Pion Exchange is *super-perturbative* in few-N systems, i.e. does *not enter before* N^{3}LO.

$\Longleftrightarrow$ *Persistence:* Footprint of Symmetries in Unitarity Limit extends far into $p_{\rm typ} \gtrsim m_\pi$, more relevant than χiral symmetry in few-N?! $\Longleftrightarrow$ Better lossless compression of Information!

Evidence: NN S-waves at N^{2}LO converge order-by-order and to PWA
inside all of **Unitarity Window** $30\ \mathrm{MeV} \lesssim k \lesssim \overline{\Lambda}_{\rm NN} \approx 300\ \mathrm{MeV}$.
Successful extension of EFT($\not\pi$) to pions.

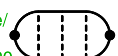
Appeal: Fine-Tuning $\Longrightarrow$ High Symmetry at Nontrivial Fixed Point:
Universality/scaling $+$ **Wigner-$\mathrm{SU}(4)$**
protected in renormalisation at FP $\Longrightarrow$ weakly broken in vicinity.
χiral symmetry not explicit at FP: less protected? $\Longrightarrow$ **Quantify!**
No Wigner in meson/1N sector $\Longrightarrow$ no change to χPT, HBχPT PC.



"Coincidence": N^{2}LO Perturbative Pions overpredict ${}^3\mathrm{SD}_1$ mixing, ${}^3\mathrm{D}_1$ $\Longrightarrow$ Zero without tensor int. at N^{2}LO.

Some Crucial Tests: If either fails without good reason, Hypothesis falsified.

N^{3}LO cf. Beane/Kaplan/Vuorinen 2009, Kaplan 2020



$\mathrm{d}\pi \to \mathrm{d}\pi$, $\gamma\mathrm{d} \to \pi\mathrm{d}$
cf. Borasoy/hg 2003

Nd scattering
cf. Bedaque/hg 2000

Nonperturbative Pions to N^{2}LO in strict perturbation LO: hg 2023

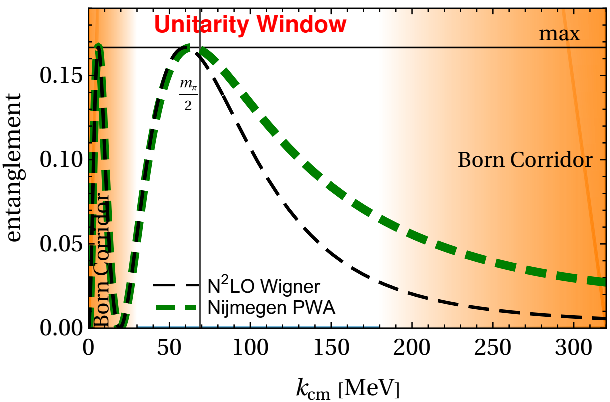
(a) What is the Small Parameter?: Entanglement? Large-N_c?

Need expansion parameter related to Wigner-$\mathrm{SU}(4)$ to characterise tensor suppression near Fixed Point.

Candidate Entanglement: Deviation from direct product position $\otimes$ spin $\otimes$ isospin.

Beane/Kaplan/Klco/Savage [1706.06550]
Farrell/Beane/... 2020-
Robin/Savage [2405.10268]

NN-scattering without higher waves & mixing: $S = \frac{1}{4}\left[\left(3\mathrm{e}^{2\mathrm{i}\delta^{^3S_1}} + \mathrm{e}^{2\mathrm{i}\delta^{^1S_0}}\right)\mathbb{1} + \vec{\sigma}_1\cdot\vec{\sigma}_2\left(\mathrm{e}^{2\mathrm{i}\delta^{^3S_1}} - \mathrm{e}^{2\mathrm{i}\delta^{^1S_0}}\right)\right]$



Unitarity Point: $S = \mathrm{e}^{2\mathrm{i}(\delta=\frac{\pi}{2})} = -\mathbb{1}$

$\Longrightarrow$ zero entanglement/classical.

Neglect higher waves & mixing:

$$\mathcal{E} = \frac{\sin^2[2(\delta^{^3S_1} - \delta^{^1S_0})]}{6} < 1 \text{ small}✓.$$

$\Longrightarrow \mathcal{E} = 0$ at Unitarity, Wigner-$\mathrm{SU}(4)$ ($\delta^{^3S_1} = \delta^{^1S_0}$).

But: $\mathcal{E} \leq \frac{1}{6}$ independent of phase shifts.

In Unitarity Window, $\mathcal{E}$ varies over full range,

and even saturates at $k \approx \frac{m_\pi}{2}$.

$\Longrightarrow$ Unitarity Window around Fixed Point irrelevant??

(a) What is the Small Parameter?: Entanglement? Large-N_c?

Need expansion parameter related to Wigner-$\mathrm{SU}(4)$ to characterise tensor suppression near Fixed Point.

Candidate Large-N_c Expansion: Kaplan/Savage [hep-ph/9509371] Kaplan/Manohar [nucl-th/9612021] Calle Cordón/Ruiz Arriola [0807.2918]

Predicts that all V_{NN} in S waves are suppressed against central (Wigner-$\mathrm{SU}(4)$) – **except** tensor ϟ.

Way out?: Wigner-$\mathrm{SU}(4)$ only realised in long-range parts, strongly broken in short-range?? Calle Cordón/Ruiz Arriola [0807.2918]

Here: Wigner-$\mathrm{SU}(4)$ breaking only in LECs: short-range – long-range ($k \to 0$) still Wigner-$\mathrm{SU}(4)$ symmetric.

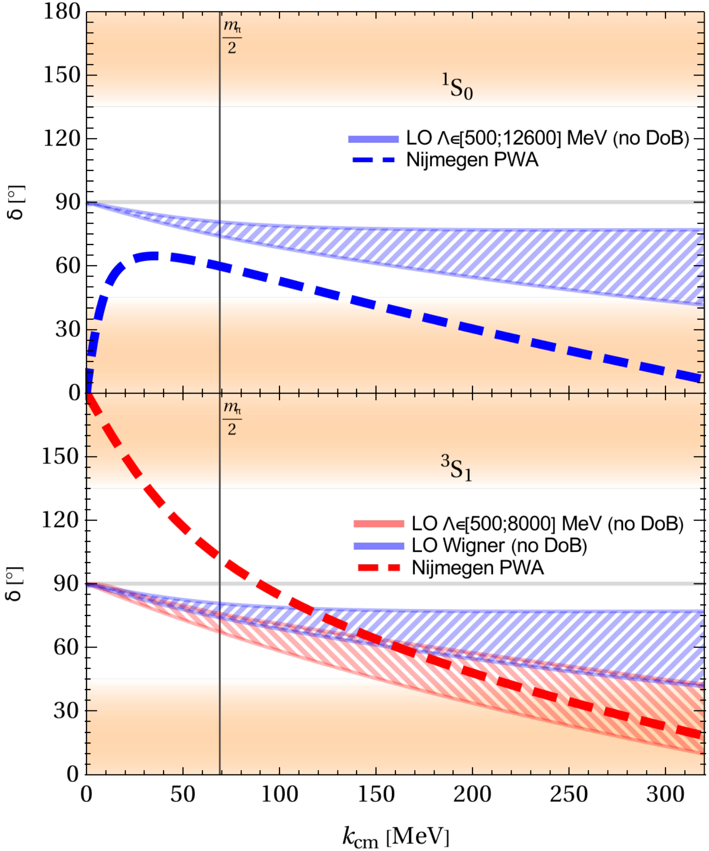
Way out?!: $1/N_c$ expansion assumes that coefficients “of natural size”.

Wigner-$\mathrm{SU}(4)$/proximity to Unitarity *forces* leading-$1/N_c$ coefficient of tensor-V_{NN} to be exact zero.

Advantage: Guaranteed to survive renormalisation by Unitarity FP symmetry.

(b) Nonperturbative Pions at LO: Maybe Not Hopeless

hg 2023, in preparation



LO, 1 mom.-indep. CT, Gaußian regulator.

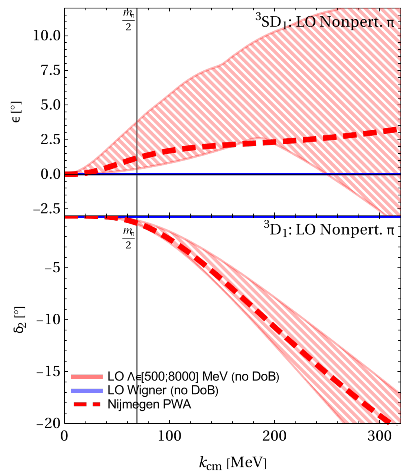
Already deviates from Unitarity $\delta = 90°$.

$\Longrightarrow$ Explicit scale breaking at LO,

$$r = \begin{cases} ^1S_0/\text{Wigner } [1\ldots2]\text{fm}; & ^3S_1\ [1.2\ldots2.5]\text{fm} \\ \text{PWA } 2.767(9)\text{fm} & 1.852(2)\text{fm} \end{cases}$$

Tensor/Wigner-breaking less compatible with unitarity than central/Wigner-symmetric.

Not Bayesian DoBs but cutoff-variation.



(c) Leading Questions

What dominates much of typical nuclear scales?

Propose controlled expansion about *Unitarity Limit*: $\frac{1}{a \to \infty} \ll k_{\text{typ}} \ll \Lambda_{\not\pi}$

("Goldilocks Point of Nuclear Physics")

$\Longrightarrow$ **Efimov effect/discrete scaling symmetry shifts focus from NN details to *single dimensionful* $3N$ *parameter* of nonperturbative renormalisation.**

Is Nuclear Physics ruled not by details of QCD or of large NN scattering, but by *proximity to* UNITARITY LIMIT*?*

Loved by Theorists since highly symmetric: scale-invariance & Wigner-$\mathrm{SU}(4)$.

PARADIGM SHIFT: De-emphasise πN & NN to see importance of $3N$ scale in Unitarity Limit.

$\Longrightarrow$ **Quantitative test of Emergence of ORDER/SIMPLICITY from complexity of Nuclear Physics.**

The Future:

König/hg/Hammer/van Kolck Kirscher/...
Kievsky/Viviani/... Drischler/...

> The efficient person gets the job done right. The effective person gets the right job done.

How well does series converge beyond NLO?

What about pions?

Test form factors, scattering,...; **nuclear levels as remnants of Efimov-like states?**: perfect isosinglet ^{6}Li,...

Falsifiable proposition by uncertainty quantification. – *Error Bars for Nuclear Theory!* –

How Far is Too Far? In heavy nuclei $\frac{B_A}{A} \approx 8\,\mathrm{MeV}$, typ. momentum $\gamma_A \sim \sqrt{2M\frac{B_A}{A}} \xrightarrow{A\to\infty} 120\,\mathrm{MeV} \begin{cases} \gg \frac{1}{a} ✓ \\ \lesssim m_\pi ?? \end{cases}$

Atomic/Molecular systems: World of "nuclear-landscape-like" states in He, Rb etc. (Feshbach resonances)?

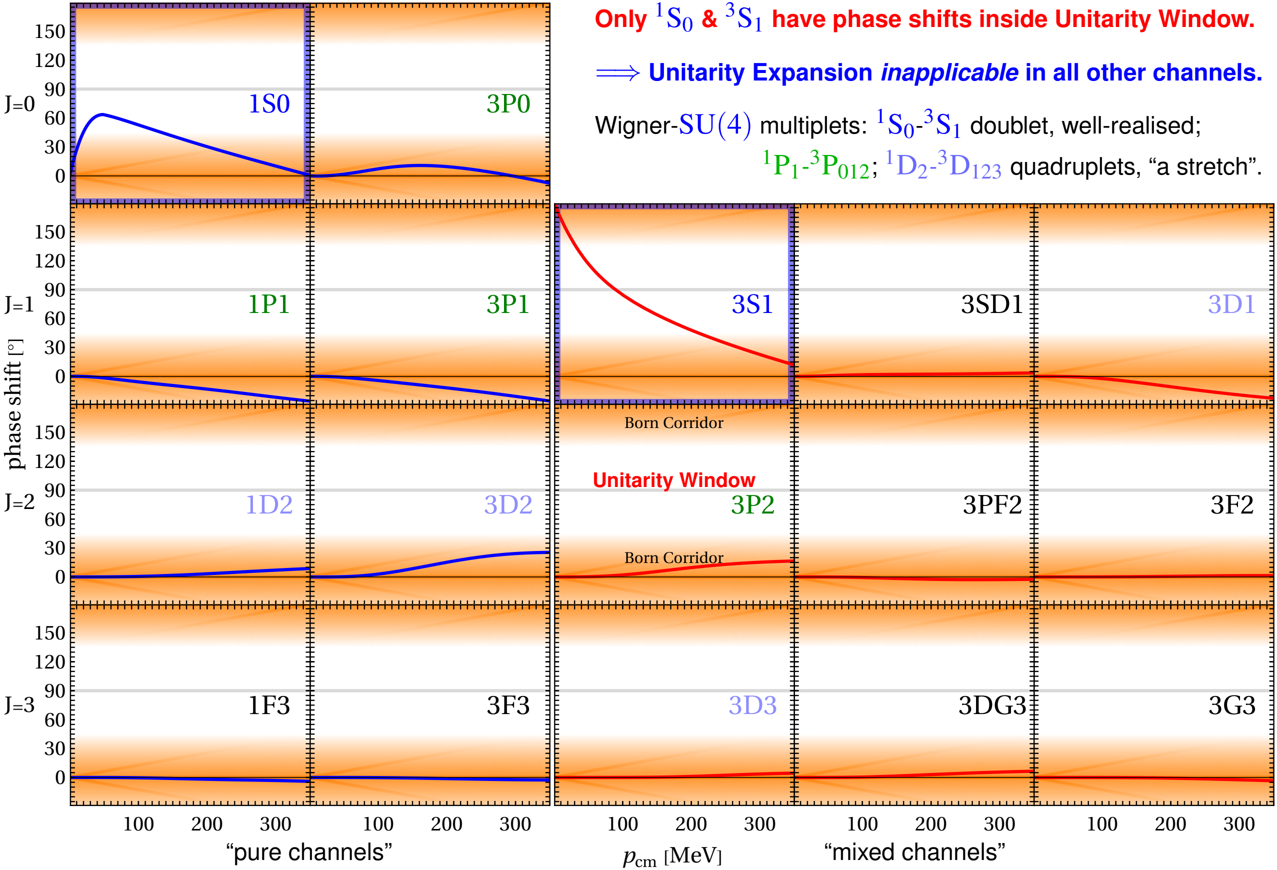
☺ You have much skill in express-
ing yourself to be effective. ☺

(a) Use Unitarity Expansion Only for Channels with Large NN Phase Shifts!

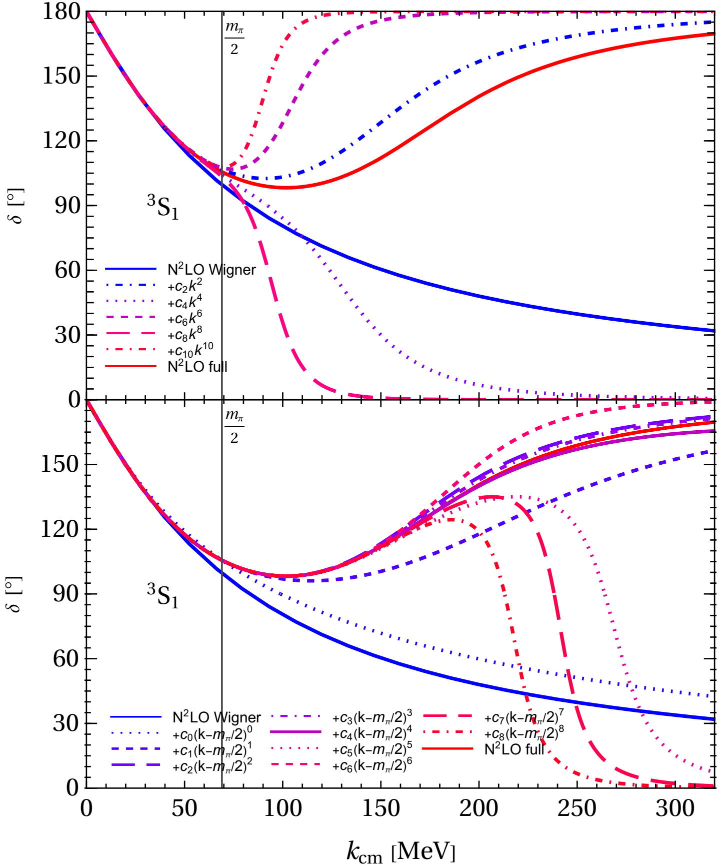
Only $^1S_0$ & $^3S_1$ have phase shifts inside Unitarity Window.

$\Longrightarrow$ **Unitarity Expansion *inapplicable* in all other channels.**

Wigner-$SU(4)$ multiplets: $^1S_0$-$^3S_1$ doublet, well-realised;
$^1P_1$-$^3P_{012}$; $^1D_2$-$^3D_{123}$ quadruplets, "a stretch".



(b) Whence the Hockey Stick in $^3S_1$?



Expand Wigner-breaking in Taylor:

$$A^{(S)}_{\text{sym}}(k) + \sum \frac{1}{n!}[A^{(S)}_{\text{break}}]^{(n)}(k_0)(k-k_0)^n$$

Expand about 0:

$k \lesssim \frac{m_\pi}{2}$: convergent, Wigner-breaking tiny

$k \gtrsim \frac{m_\pi}{2}$: no convergence

$\Longrightarrow$ ERE not the problem.

Sorry, no Cohen/Hansen 1999.

Expand about 1st branch point scale $\frac{m_\pi}{2}$:

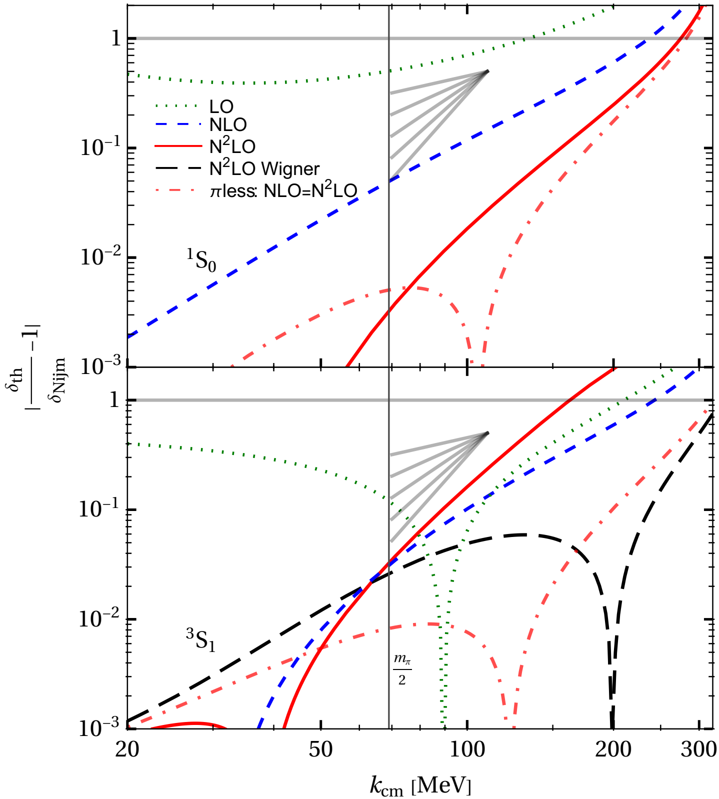
$k \lesssim \frac{m_\pi}{\sqrt{2}}$: convergent, Wigner-breaking tiny

(larger distance to branch point)

$k \lesssim \frac{3}{2}m_\pi$ (>2nd br. pt. scale): convergent

$k \gtrsim \frac{3}{2}m_\pi$: asymptotic (optimal: incl. k^4)

(c) Convergence to Data



$$\frac{\delta(\mathrm{N}^n\mathrm{LO})-\delta(\mathrm{PWA})}{\delta(\mathrm{PWA})}\sim\left(\frac{k,m_\pi}{\overline{\Lambda}}\right)^{n+1}$$

at $\mathrm{N}^n\mathrm{LO}$ with empirical breakdown scale $\overline{\Lambda}$.

$^1\mathrm{S}_0$ and Wigner-symmetric $^3\mathrm{S}_1$:

consistent slopes and

$\overline{\Lambda}\approx 270\;\mathrm{MeV}\approx\overline{\Lambda}_{\mathrm{NN}}$ OPE scale.

Full $^3\mathrm{S}_1$:

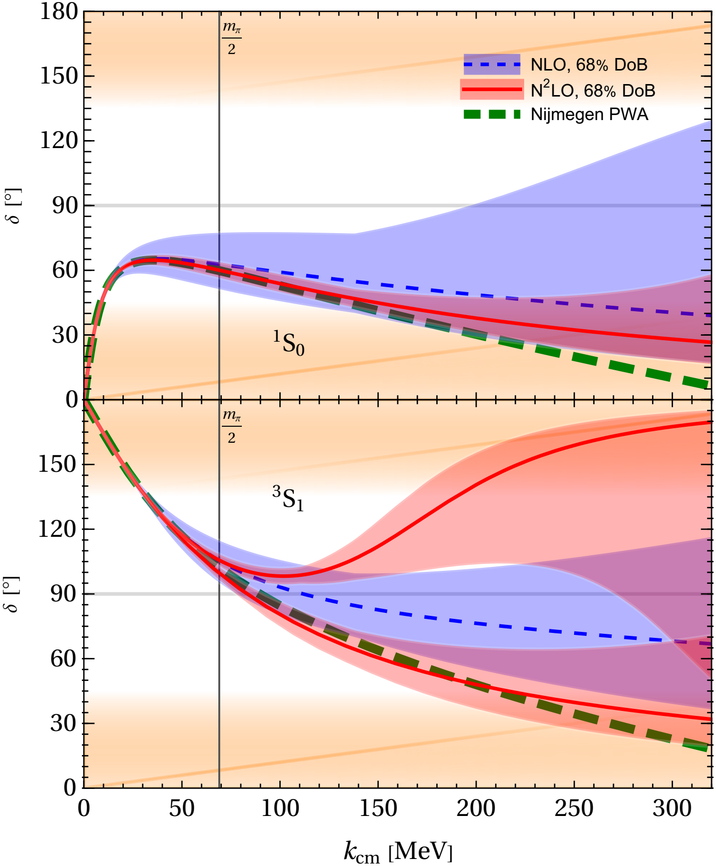
$\mathrm{N}^2\mathrm{LO}$ worse than NLO for $\gtrsim 70\;\mathrm{MeV}$.

Picture obscured by points where

theory & PWA identical (“artificial zero”),

or PWA close to zero (“artificial ∞”).

(d) NLO & N^{2}LO Bayesian Truncation Uncertainties



Apply "$\max$" criterion to $\cot\delta$ order-by-order:

Unitarity: $k\cot\delta_{LO} = 0 \Rightarrow$ "$-ik$" sets scale.

Bayesian N^{2}LO truncation uncertainty at k:

$$\pm Q^3 \max\left\{\frac{\cot\delta_0(k)-\cot\delta_0(0)}{Q};\frac{\cot\delta_1(k)}{Q^2}\right\}$$

with $Q = \dfrac{\max\{k;m_\pi\}}{\overline{\Lambda}_{NN}\sim 300\ \text{MeV}}$

NLO: rescaled to 68% DoB,

assuming uniform&log-uniform prior.

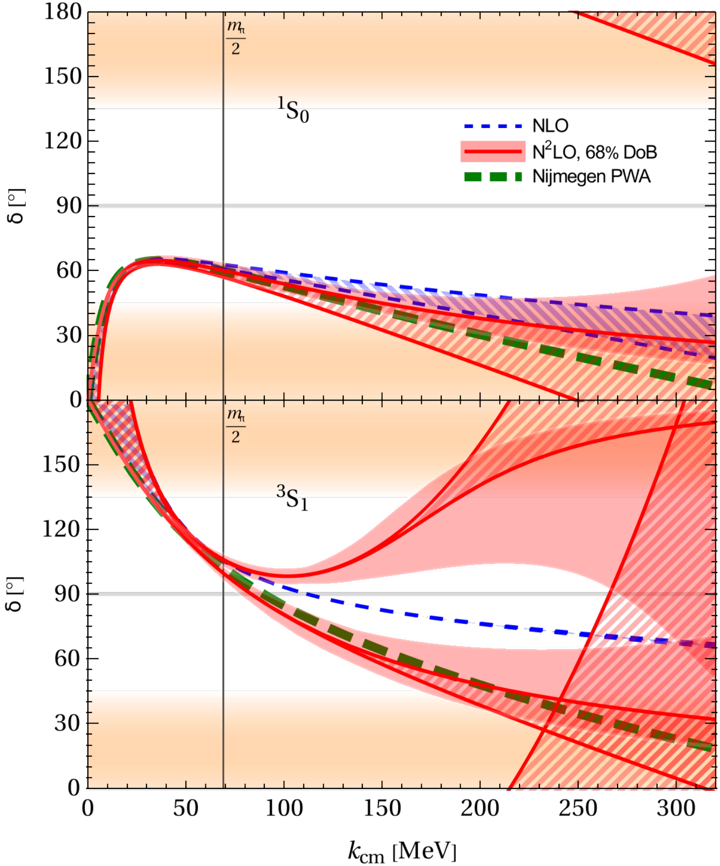
Only Wigner-symmetric forms have

N^{2}LO uncertainties consistent with NLO,

and NLO&N^{2}LO consistent with PWA.

(e) Different Ways To Extract Phase Shifts at NLO and N^{2}LO

Teng/hg [2410.09653]



So far:

$$\text{kcot}\delta = 0_{\text{LO}} + \text{kcot}\delta_{\text{NLO}} + \text{kcot}\delta_{\text{N}^2\text{LO}}$$
$$= -\frac{4\pi}{M}\left[\frac{A_0}{A_{-1}^2} - \frac{A_0^2}{A_{-1}^3} + \frac{A_1}{A_{-1}^2}\right]$$

is fundamental, derive $\delta(k)$ from it.

$$\stackrel{k\to 0}{\longrightarrow} 0_{\text{LO}} + \left(-\frac{1}{a} + \frac{r}{2}k^2\right)_{\text{N}^{1+2}\text{LO}} + \mathcal{O}(k^4)$$

constructed to reproduce ERE.

Hatched: difference to

directly from amplitude KSW 1999,FMS 2000

$$\delta(k) = \frac{\pi}{2}\Big|_{-1} + \left(\frac{1}{ka} - \frac{kr}{2}\right)\Big|_0 + \dots$$

$\to \infty$ for $k \to 0$ outside Unitarity Window.

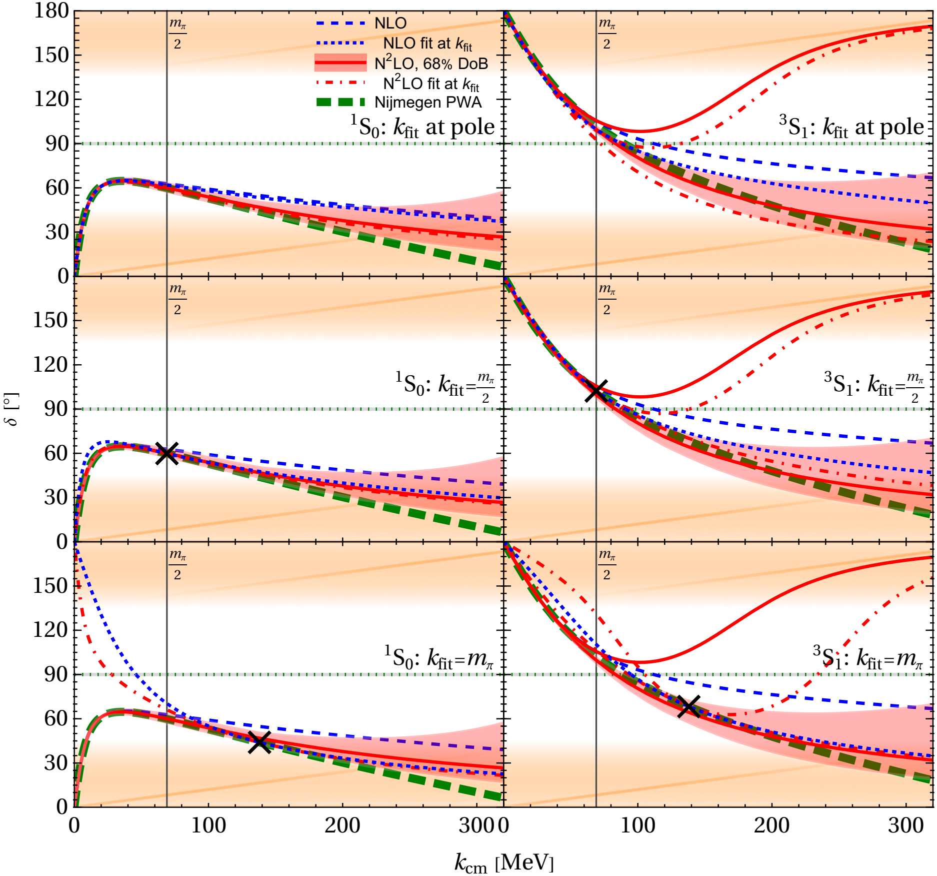
Methods agree inside Unitarity Window

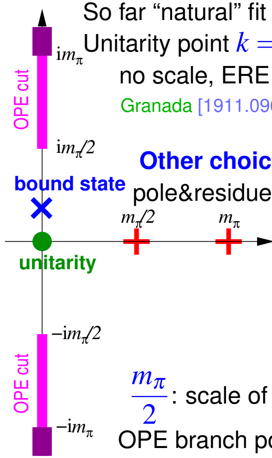
$\frac{1}{ka}, \frac{kr}{2} < 1$ (must in centre $|\cot\delta| \to 0$):

Independent assessment of

truncation uncertainty, consistent with Bayes.

(f) Different Renormalisation/Parameter-Determination Points





So far "natural" fit at Unitarity point $k=0$: no scale, ERE

Granada [1911.09637]

Other choices:

pole&residue: ok

$\frac{m_\pi}{2}$: scale of 1st OPE branch point

No cure to hockey-stick.

Uncertainties & breakdown scale very similar.

m_π: 2nd OPE branch point

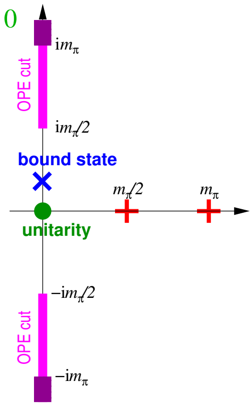
No cure to hockey-stick.

Low-k $^1S_0$ bad inside Unitarity Window.

(g) Virtual/Real Bound-State Pole Positions and Residues

$$\frac{1}{A_{-1}(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+A_0(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+A_1(\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots)+\dots} \stackrel{!}{=} 0$$

$$\frac{1}{Z} = \mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}k}\,(k\cot\delta(k)-\mathrm{i}k)\Big|_{k=\mathrm{i}\gamma_{-1}+\mathrm{i}\gamma_0+\mathrm{i}\gamma_1+\dots}$$



For $k_{\text{fit}}=0$, pions cannot correct a, r since we force the ERE values Granada [1911.09637]

$\Longrightarrow$ pole at binding momentum $\mathrm{i}\gamma = \frac{\mathrm{i}}{a}\left(1+\frac{r}{2a}+\frac{r^2}{2a^2}+\mathcal{O}(\frac{r^3}{a^3})\right)$

with residue $Z = 1+\frac{r}{a}+\frac{3r^2}{2a^2}+\mathcal{O}(\frac{r^3}{a^3})$.

For general k_{fit}, match to $k\cot\delta_{\text{PWA}}(k_{\text{fit}})$, $\frac{\mathrm{d}}{\mathrm{d}k}k\cot\delta_{\text{PWA}}(k_{\text{fit}})$. $\Longrightarrow$ Predict a, r.

k_{fit}	$^1S_0$ scatt. length a [fm]	$^1S_0$ eff. range r [fm]	$^1S_0$ (bind. mom.,residue) $(\gamma$ [MeV]$,Z)$	$^3S_1$ scatt. length a [fm]	$^3S_1$ eff. range r [fm]	$^3S_1$ (bind. mom.,residue) $(\gamma$ [MeV]$,Z)$
ERE	$-23.735(6)*$	$2.673(9)*$	$(-7.892, 0.9034)$	$5.435(2)*$	$1.852(2)*$	$(+47.7023, 1.689)*$
pole	-23.7104	2.7783		5.6128	2.3682	
$\frac{m_\pi}{2}$ NLO	-38.988	3.3270	$(-4.86, 0.925)$	4.9310	2.4966	$(+55., 1.9)$
$\frac{m_\pi}{2}$ N^{2}LO	-25.428	2.7281	$(-7.34, 0.910(2))$	4.7768	2.4492	$(+57(3)., 1.9(2))$
$\frac{m_\pi}{2}$ sym.				5.4625	1.6124	$(+43.0(5), 1.42(4))$
m_π NLO	$+\ 9.2856$	4.2285	$(+28., 1.8)$	$3.3442\dagger$	$3.1886\dagger$	$(+114., 3.)$ ϟ
m_π N^{2}LO	$+34.3335$	2.8956	$(+6.01, 1.10)$	$1.8376\dagger$	$3.3741\dagger$	$(+387(330)., 7(9).)$ ϟ
m_π sym.				4.5344	1.7006	$(+54(1)., 1.5(1))$

Bayesian N^{2}LO uncertainties $\quad *$: input $\quad$ ϟ: cannot converge: $Q\sim\frac{1}{ka},\frac{kr}{2}\ll 1 \Longrightarrow \frac{r}{2a}\ll 1$ ϟ