Adiabatic Hydrodynamization and the Emergence of Attractors

Heavy Ion Physics in the EIC Era
Institute for Nuclear Theory, University of Washington
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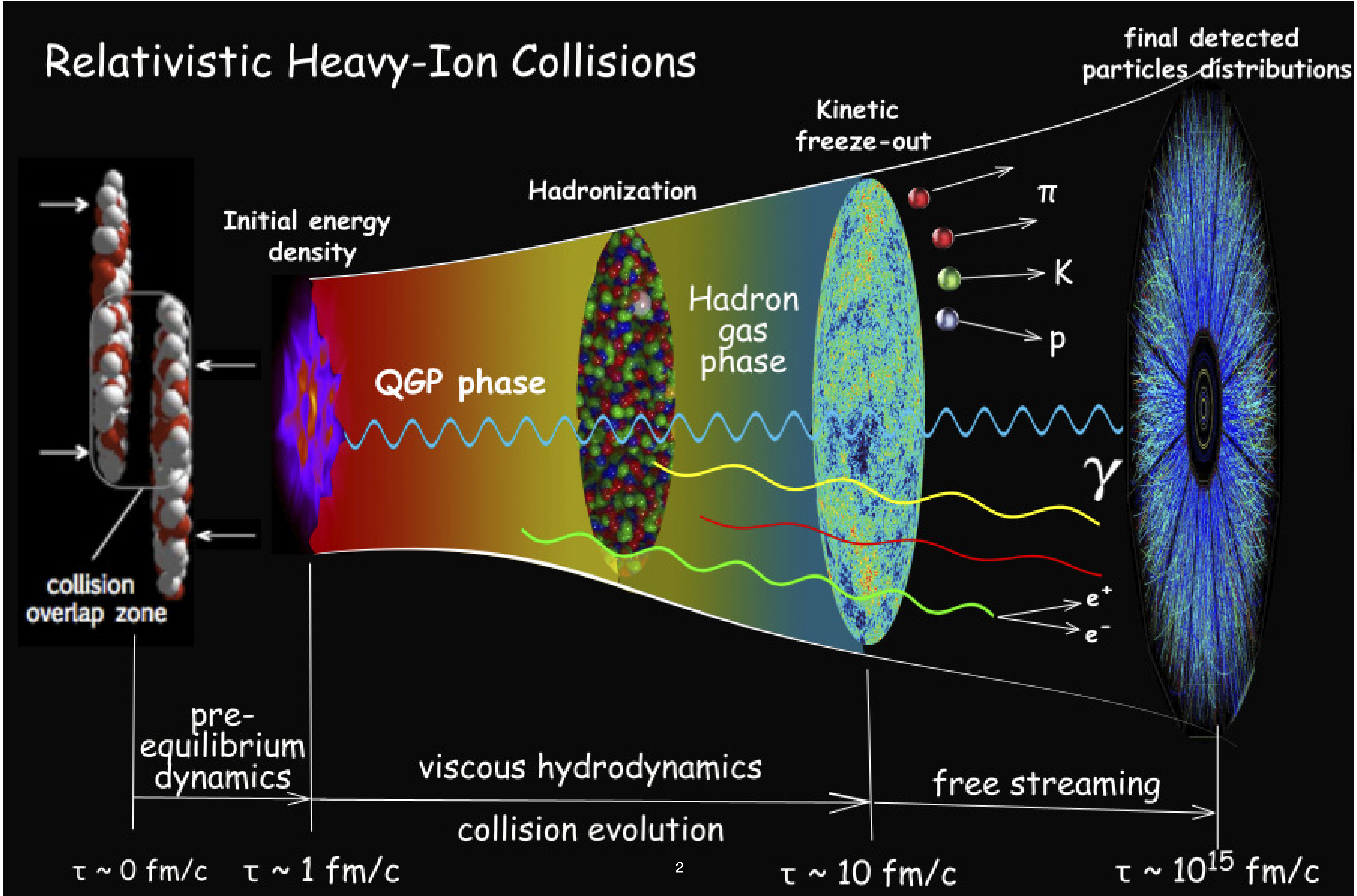
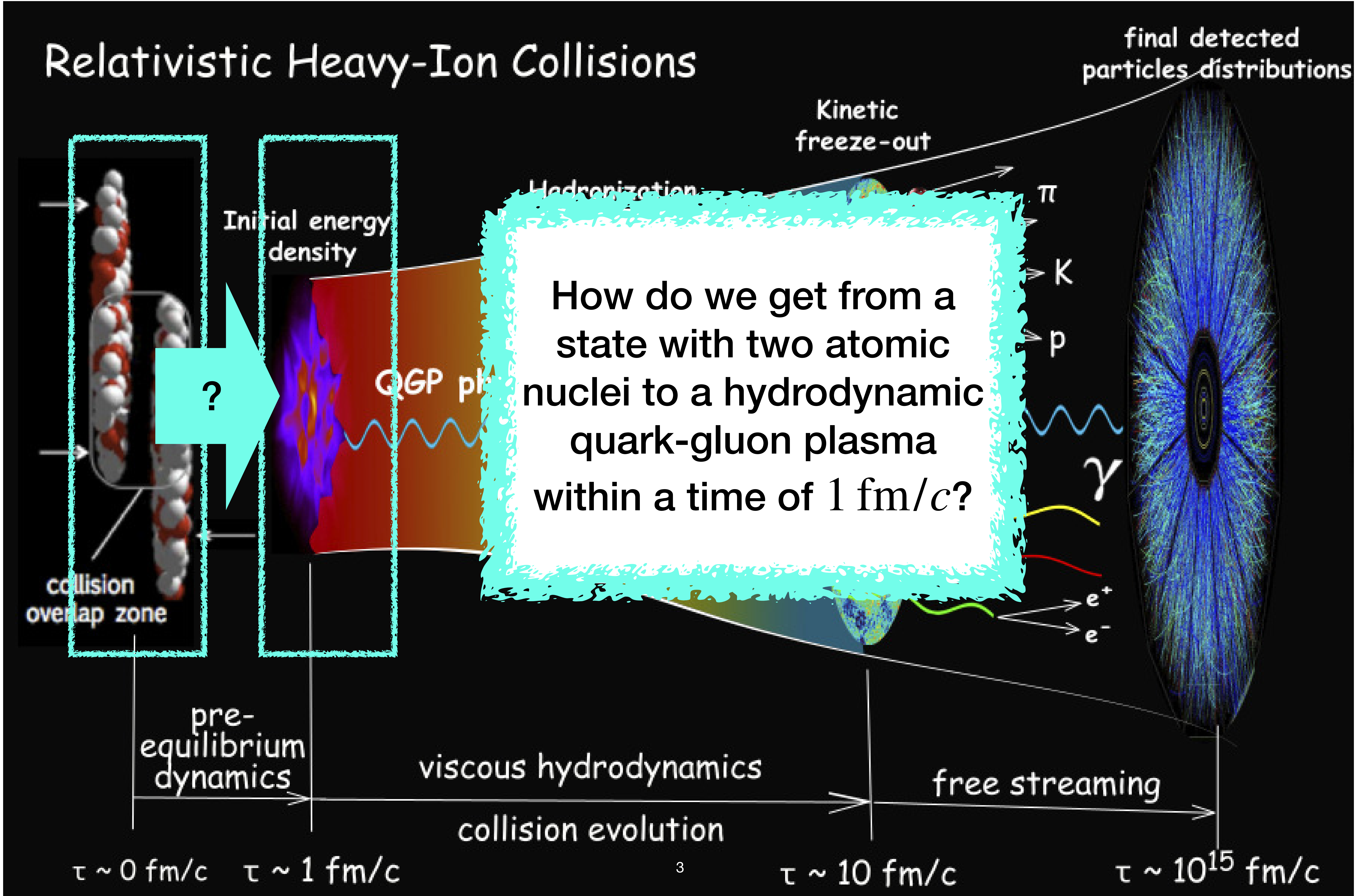
Bruno Scheihing Hitschfeld
based on 2203.02427 and 2405.17545



credit: Paul Sorensen and Chun Shen



How do we get from a state with two atomic nuclei to a hydrodynamic quark-gluon plasma within a time of $1\,\mathrm{fm}/c$?

credit: Paul Sorensen and Chun Shen

How is the memory of the initial condition lost?

A bit of history

understanding hydrodynamization in HICs

- In the early 2000's, it was realized that data from the Relativistic Heavy Ion Collider (RHIC) on hadron spectra and elliptic flow could be described by hydrodynamics starting at $\tau = 0.6\,\mathrm{fm}/c$ after the collision.

  Ulrich W. Heinz, Peter F. Kolb, *"Early thermalization at RHIC," Nucl. Phys. A* 702 (2002) 269-280

- The first calculations based on a microscopic theory that explained why this could happen so rapidly were done in strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory via the AdS/CFT correspondence.

  Paul M. Chesler, Laurence G. Yaffe, *"Horizon formation and far-from-equilibrium isotropization in supersymmetric Yang-Mills plasma," Phys. Rev. Lett.* 102 (2009) 211601

- In the last decade it was shown that rapid hydrodynamization in HICs could be described within the Effective Kinetic Theory of weakly coupled QCD.

  Aleksi Kurkela, Yan Zhu, *"Isotropization and hydrodynamization in weakly coupled heavy-ion collisions," Phys. Rev. Lett.* 115 (2015) 18, 182301

Out of equilibrium attractors

emergence of universal behavior — loss of memory

- Many theories describing the pre-hydrodynamic stage exhibit so-called “attractor” solutions. These solutions have been sought, found, and intensively studied over the past decade.
- The nature of the attractors can be different in different models:

A. Kurkela, W. van der Schee, U. A. Wiedemann, B. Wu “What attracts to attractors?” Phys. Rev. Lett. 124, 102301 (2020)

How do we write such a theory?

- For today, let me focus on kinetic theories:

$$\partial_t f = - C[f]$$

  - $f = f(\mathbf{x}, \mathbf{p}, t)$ is the particle number per unit density per unit momentum.
  - Describes interacting quantum many-body theories with weakly-coupled quasiparticles. Interactions are described by the collision kernel $C[f]$.
  - Allows for nontrivial initial states & provides a description of thermalization.
- Challenges for the future: strongly coupled theories.

How do we identify long-lived modes?

How do we identify attractors?

- A kinetic equation $\partial_t f = -C[f]$ is first-order in time derivatives, just like a Schrödinger equation:

$$\partial_t \psi = -i\mathscr{H}\psi$$

- The parallel becomes clear if we are able to write the kinetic equation as

$$\partial_t f = -H[f]\,f\,,$$

because then we can study $H[f]$ as a generator of time evolution.

- To make the discussion more familiar, take $H[f] \longrightarrow H(\tau)$.

Adiabatic hydrodynamization (AH)

Adiabatic hydrodynamization

as proposed by Brewer, Yan, and Yin

- Idea: the essential feature of an attractor is a reduction in the number of quantities needed to describe the system.

- Brewer, Yan and Yin conjectured that this is due to an emergent timescale $\tau_{\mathrm{Redu}} \ll \tau_{\mathrm{Hydro}}$ after which a set of "pre-hydrodynamic" slow modes (that gradually evolve into hydrodynamic modes) govern the system.

- Their proposal: try to understand the emergence of τ_{Redu} (at which only slow modes remain) using the machinery of the adiabatic approximation in quantum mechanics.

J. Brewer, L. Yan, Y. Yin "Adiabatic hydrodynamization in rapidly-expanding quark-gluon plasma" Phys. Lett. B 816, 136189 (2021)

Adiabatic hydrodynamization

adiabatic theorem and the notion of adiabaticity

- Consider a system whose evolution is given by

$$\partial_\tau |\psi\rangle = -H(\tau)|\psi\rangle,$$

where $H(\tau)$ has eigenstates/eigenvalues $\{|n(\tau)\rangle, E_n(\tau)\}_{n=0}^{\infty}$:

$$H(\tau)|n(\tau)\rangle = E_n(\tau)|n(\tau)\rangle.$$

- Then, one may write the system's evolution as

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^\tau E_n(\tau')d\tau'} |n(\tau)\rangle.$$

- Adiabaticity is the degree to which transitions between different instantaneous eigenstates are suppressed:

$$\text{Adiabaticity for the } n\text{-th eigenstate} \iff \frac{\dot{a}_n}{a_n} \ll |E_n - E_m|, \text{ for } n \neq m.$$

- When this is the case, provided there is an "energy" gap between the ground state and the excited states, one has

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle$$
$$\approx a_0\, e^{-\int^{\tau} E_0(\tau')d\tau'} |0(\tau)\rangle\,,$$

that is to say, the dynamics of the system collapses onto a single form.

$\Longrightarrow$ Reduction in the number of variables needed to describe the system.

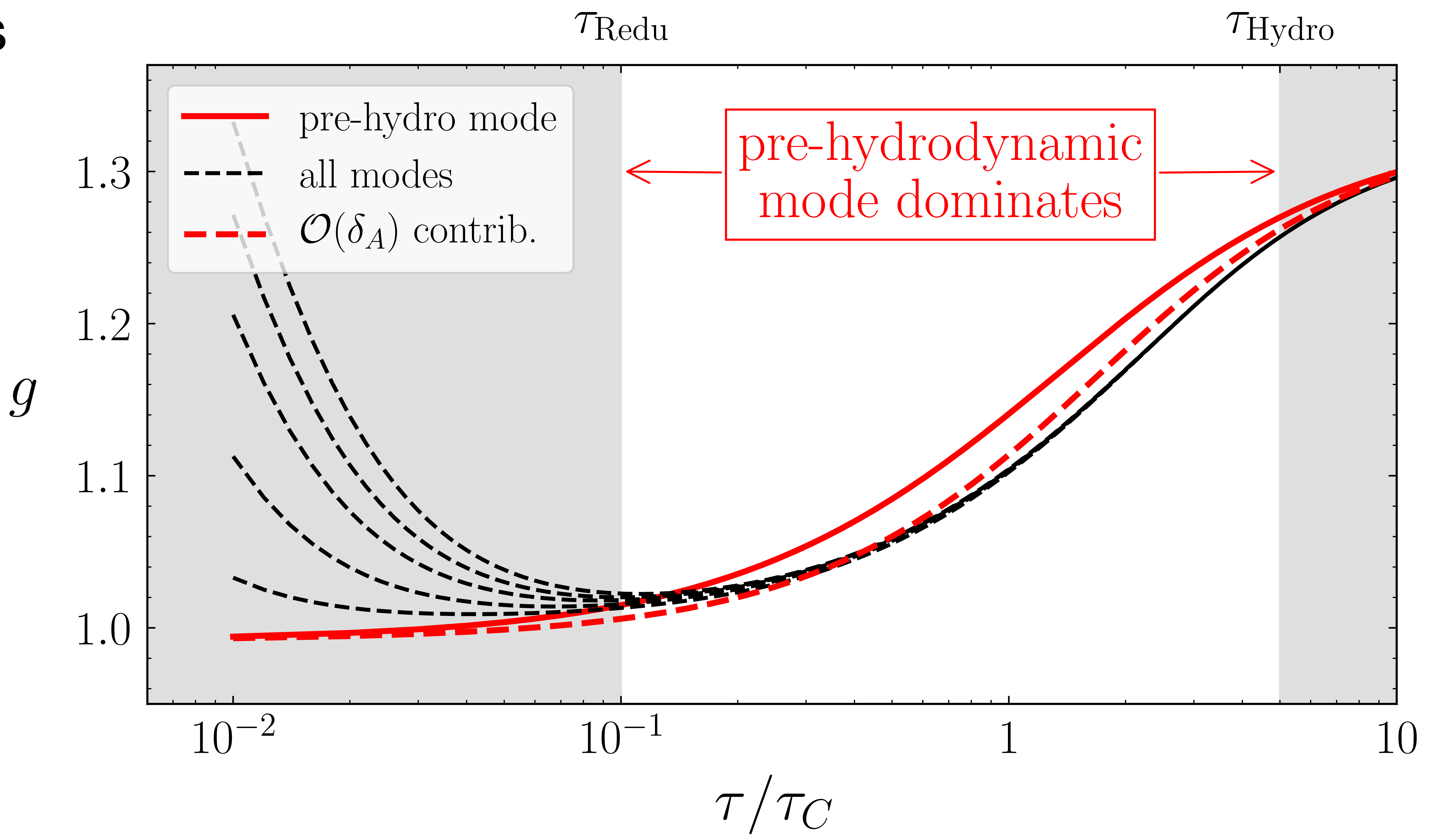
Adiabatic hydrodynamization

Brewer, Yan, and Yin's RTA analysis

- The first exploration of this hypothesis was made by studying an RTA kinetic theory in a Bjorken-expanding plasma:

$$\partial_\tau f(\mathbf{p},\tau) - \frac{p_z}{\tau}\partial_{p_z} f(\mathbf{p},\tau) = -\frac{f(\mathbf{p},\tau) - f_{\text{eq}}(\mathbf{p}; T(\tau))}{\tau_C}$$

$$g(\tau) = \partial_{\ln\tau} \ln \epsilon(\tau)$$



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‘Bottom-up’ thermalization

R. Baier, A. H. Mueller, D. Schiff, D. T. Son, “‘Bottom-up’ thermalization in heavy ion collisions” Phys. Lett. B 502, 51-58 (2001)

‘Bottom-up’ thermalization
as formulated by Baier, Mueller, Schiff, and Son

In the BMSS scenario (in weakly-coupled QCD), thermalization proceeds as

1. Over-occupied hard gluons $f_g \gg 1$ at very early times $1 \ll Q_s\tau \ll \alpha_s^{-3/2}$
2. Hard gluons become under-occupied $f_g \ll 1$, when $\alpha_s^{-3/2} \ll Q_s\tau \ll \alpha_s^{-5/2}$
3. Thermalization of the soft sector after $\alpha_s^{-5/2} \ll Q_s\tau$

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Specifically, stage 1 predicts that

$$\gamma \equiv -\frac{1}{2}\frac{\partial_{\ln\tau}\langle p_z^2\rangle}{\langle p_z^2\rangle} = \frac{1}{3}\,,\quad \beta \equiv -\frac{1}{2}\frac{\partial_{\ln\tau}\langle p_\perp^2\rangle}{\langle p_\perp^2\rangle} = 0\,.$$

R. Baier, A. H. Mueller, D. Schiff, D. T. Son, "‘Bottom-up’ thermalization in heavy ion collisions" Phys. Lett. B 502, 51-58 (2001)

The gluon collision kernel
in the elastic small-angle scattering approximation

- During the earliest stages of the hydrodynamization process of a weakly coupled gluon gas, it is appropriate to work in the small-angle scattering approximation:

$$\partial_\tau f - \frac{p_z}{\tau}\partial_{p_z} f = 4\pi\,\alpha_s^2 N_c^2 l_{\text{Cb}}[f]\Big[I_a[f]\,\nabla_{\mathbf{p}}^2 f + I_b[f]\,\nabla_{\mathbf{p}}\cdot\big(\hat{p}(1+f)f\big)\Big],$$

where

$$I_a[f] = \int_{\mathbf{p}} (1+f)f\,,\quad I_b[f] = \int_{\mathbf{p}} \frac{2}{p} f = \frac{m_D^2}{2N_c g_s^2}\,,\quad l_{\text{Cb}}[f] = \ln\left(\frac{p_{\text{UV}}}{p_{\text{IR}}}\right) \approx \frac{1}{2}\ln\left(\frac{\langle p_\perp^2\rangle}{m_D^2}\right)$$

A.H. Mueller, “The Boltzmann equation for gluons at early times after a heavy ion collision,” Phys. Lett. B 475, 220 (2000)

- Furthermore, for the first stage of the bottom-up scenario we can consider the approximations: J. Brewer, B. Scheihing-Hitschfeld, Y. Yin "Scaling and adiabaticity in a rapidly expanding gluon plasma" JHEP 05 (2022) 145

$$\frac{\langle p_z^2 \rangle}{\langle p_\perp^2 \rangle} \ll 1 \,, \quad f \gg 1 \,,$$

with which the kinetic equation simplifies to

$$\partial_\tau f - \frac{p_z}{\tau} \partial_{p_z} f = 4\pi \, \alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f] \, \nabla_{\mathbf{p}}^2 f \,.$$

How does adiabaticity come into play?

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle$$

$$\overset{?}{\approx} a_0\, e^{-\int^{\tau} E_0(\tau')d\tau'} |0(\tau)\rangle \,.$$

Before that: what is $|\psi\rangle$? What is $|0(\tau)\rangle$?

Scaling and adiabaticity

[1] J. Brewer, B. Scheihing-Hitschfeld, and Y. Yin, *Scaling and adiabaticity in a rapidly expanding gluon plasma*, *JHEP* **05** (2022) 145, [arXiv:2203.02427].

Typical time evolution of the gluon occupation number in a weakly-coupled Bjorken-expanding plasma

The adiabatic frame

connecting scaling and adiabaticity

We have an equation of the form $\partial_\tau f = -H(\tau) f$. Two options:

1. Find the instantaneous eigenstates of $H(\tau)$ and see if the adiabatic criterion is satisfied, or

2. Introduce a new “frame” that optimizes adiabaticity:

$$f(p_\perp, p_z, \tau) = A(\tau)\, w\Big(p_\perp / B(\tau), p_z / C(\tau); y(\tau)\Big)$$

with a new distribution function $w(\zeta, \xi; y)$ and rescaled coordinates $\zeta = p_\perp / B(\tau)$, $\xi = p_z / C(\tau)$, $y = \ln(\tau/\tau_I)$. Then, we have a new Hamiltonian $\mathscr{H}$:

$$\partial_y w = -\mathscr{H}(y) w.$$

What is the advantage of this?

- Because A, B, C are a choice of coordinates (a "gauge" choice to describe the system), we can choose them to simplify the time dependence of $\mathcal{H}$.
- Then, we [1] find that at early times it is possible to write

$$\mathcal{H} = \alpha - (1-\gamma)\left[\partial_\xi^2 + \xi\,\partial_\xi\right] + \beta\left[\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta + \zeta\,\partial_\zeta\right],$$

$$\alpha = \frac{\partial_y A}{A}$$

$$\beta = -\frac{\partial_y B}{B}$$

$$\gamma = -\frac{\partial_y C}{C}$$

which is a separable Hamiltonian of the form

$$\mathcal{H} = f_0(y)\,H_0 + f_1(y)\,H_\xi + f_2(y)\,H_\zeta,$$

where the Hamiltonians H_0, H_ξ, H_ζ are constant and can be "diagonalized" simultaneously. In this situation, the adiabatic approximation is exact.

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- The eigenvalues of $\mathscr{H}$ are $\mathscr{E}_{n,m} = 2n(1-\gamma) - 2m\beta\ , \quad n, m = 0, 1, 2, \ldots$

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$$\mathscr{H} = \alpha - (1 - \cdots \frac{1}{\zeta}\partial_\zeta + \zeta\,\partial_\zeta \Big],$$

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$$\mathscr{H} = f_0 \cdots H_\zeta,$$

where the Hamiltonians H_0, H… …an be "diagonalized" simultaneously. In this situati… …nation is exact.

Gapped energy levels!

$\Longrightarrow$ Ground state will dominate after a transient time

Also: no need for γ, β to have reached their asymptotic values

- The eigenvalues of $\mathscr{H}$ are $\mathcal{E}_{n,m} = 2n(1-\gamma) - 2m\beta$, $\quad n, m = 0, 1, 2, \ldots$

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That is to say,

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle$$

$$\to a_0\, e^{-\int^{\tau} E_0(\tau')d\tau'} |0(\tau)\rangle\ .$$

$\Longrightarrow$ the beginning of the hydrodynamization process in a (weakly coupled) HIC proceeds through the dominance of low-energy state(s).

What about the dynamics of the “frame” variables?

$$\partial_y A = \alpha A\,, \quad \partial_y B = -\beta B\,, \quad \partial_y C = -\gamma C$$

$q = 4\pi\alpha_s^2 N_c^2 l_{\text{Cb}}[f] I_a[f]\tau$

Flow of γ, β under time evolution

Open circles: fixed points with $\dot{l}_{\text{Cb}} = 0$, Filled circles: fixed points with $\dot{l}_{\text{Cb}} = 0.4$

$\text{over} - \text{occupied}\ (A \gg 1 \iff ''f \gg 1'')$:

$$\partial_y \beta = \left(\gamma + 4\beta - 1 + \dot{l}_{\text{Cb}}\right)\beta\,,$$

$$\partial_y \gamma = \left(3\gamma + 2\beta - 1 + \dot{l}_{\text{Cb}}\right)(\gamma - 1)\,.$$

$\text{dilute}\ (A \ll 1 \iff ''f \ll 1'')$:

$$\partial_y \beta = \left(2\beta + \dot{l}_{\text{Cb}}\right)\beta\,,$$

$$\partial_y \gamma = \left(2\gamma + \dot{l}_{\text{Cb}}\right)(\gamma - 1)\,.$$

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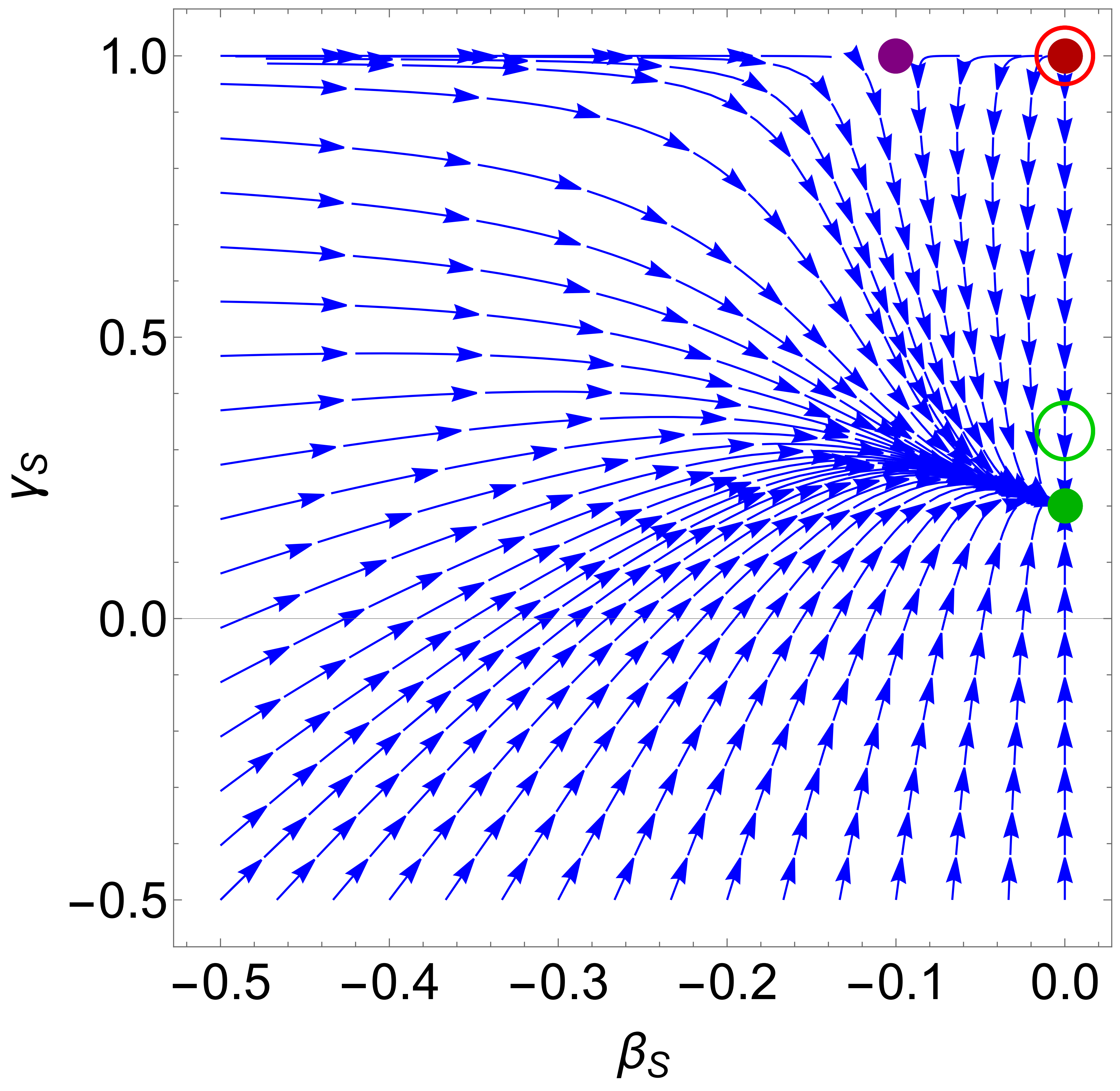

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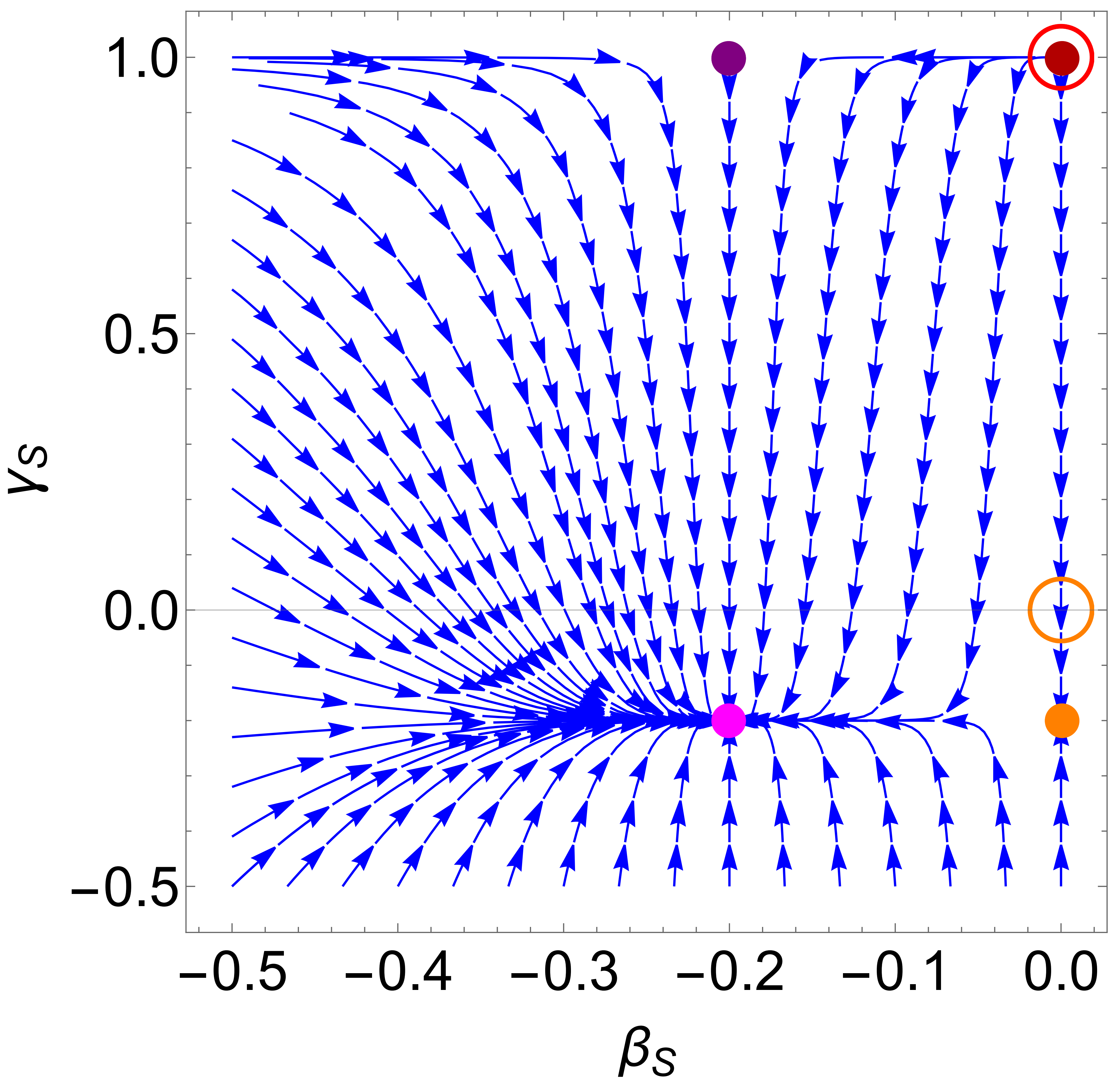


[6] A. Mazeliauskas, J. Berges, "Prescaling and far-from-equilibrium hydrodynamics in the quark-gluon plasma" Phys. Rev. Lett. 122, 122301 (2019)
see also [7] A. N. Mikheev, A. Mazeliauskas, J. Berges, "Stability analysis of nonthermal fixed points in longitudinally expanding kinetic theory" Phys. Rev. D **105**, 116025 (2022)

Scaling exponents

comparison with QCD EKT

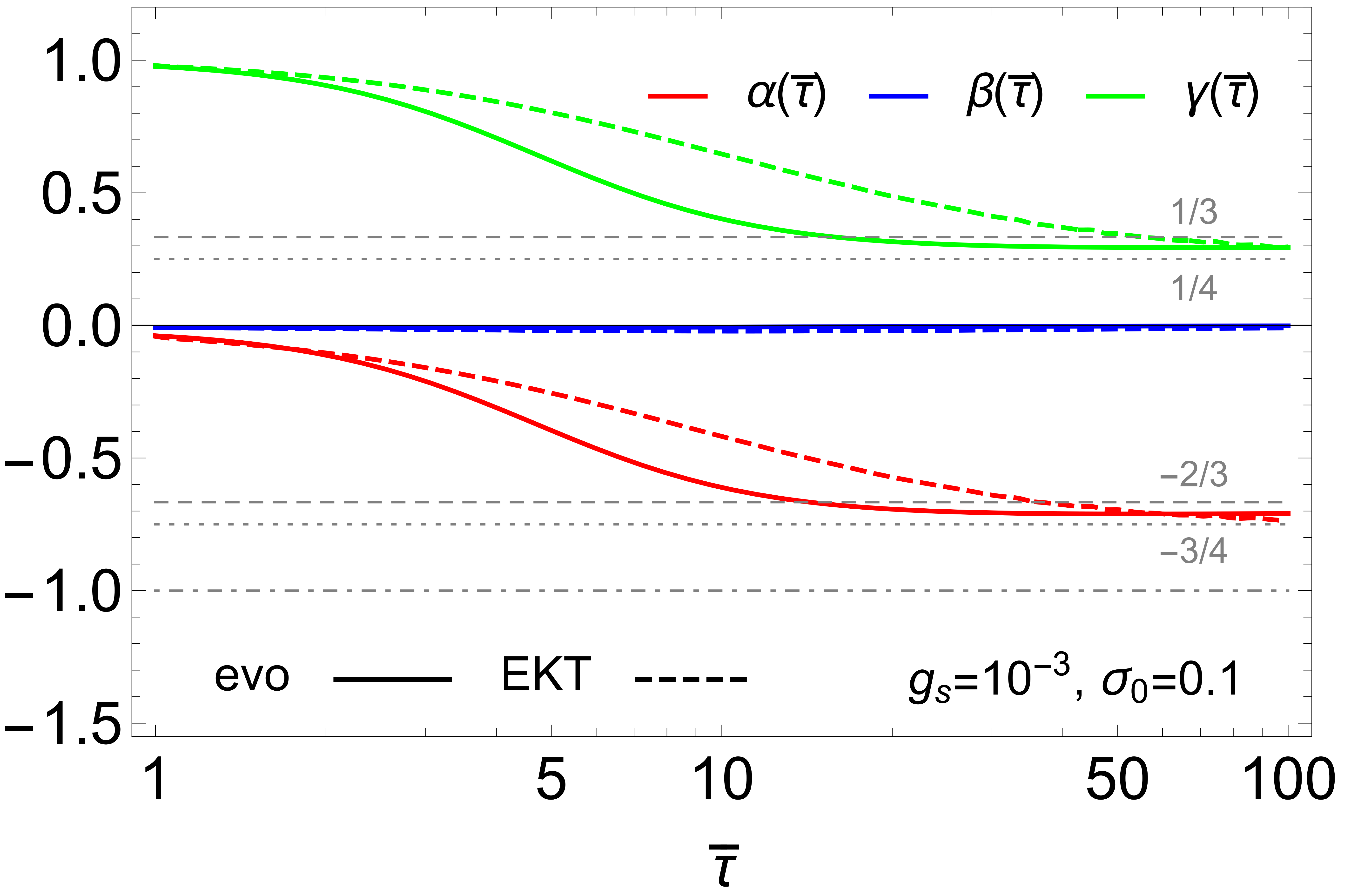
- We compare our results with those of [6], using the same initial condition:

$$f(\tau_I) = \frac{\sigma_0}{g_s^2} \exp\left(-\frac{p_\perp^2 + \xi^2 p_z^2}{Q_s^2} \right).$$

- In our description, for this initial condition we predict a deviation from the BMSS scaling exponents given by:

$$\delta\gamma \equiv \gamma - \frac{1}{3} = -\frac{1}{3\ln\left(\frac{4\pi\tau}{N_c \tau_I \sigma_0}\right)}$$

Flow equations [1] (solid) versus QCD EKT [6] (dashed)



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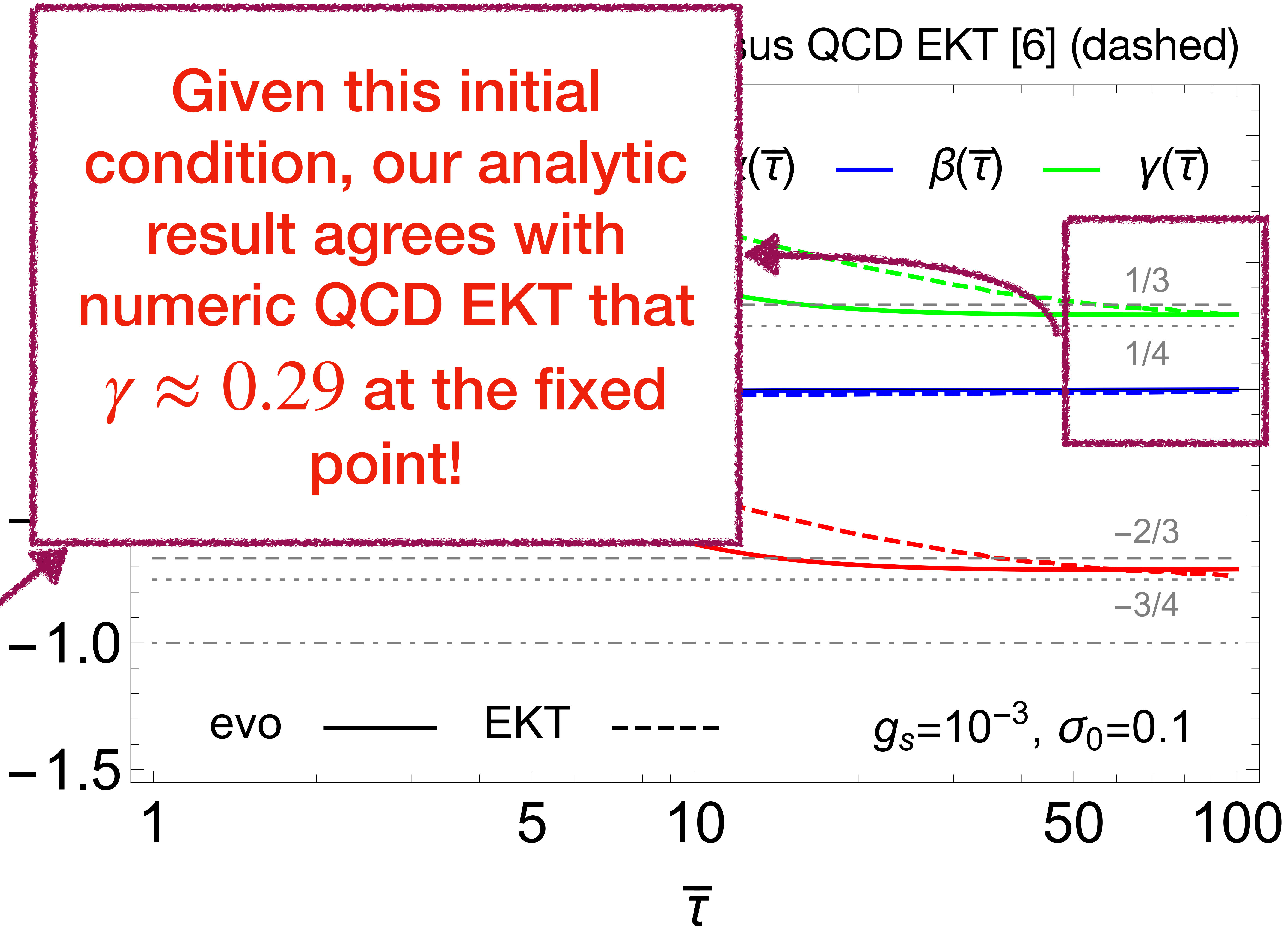


Given this initial condition, our analytic result agrees with numeric QCD EKT that $\gamma \approx 0.29$ at the fixed point!

Typical time evolution of the gluon occupation number in a weakly-coupled Bjorken-expanding plasma

Approach to Hydrodynamics

[2] K. Rajagopal, B. Scheihing-Hitschfeld, and R. Steinhorst, *Adiabatic Hydrodynamization and the Emergence of Attractors: a Unified Description of Hydrodynamization in Kinetic Theory*, arXiv:2405.17545.

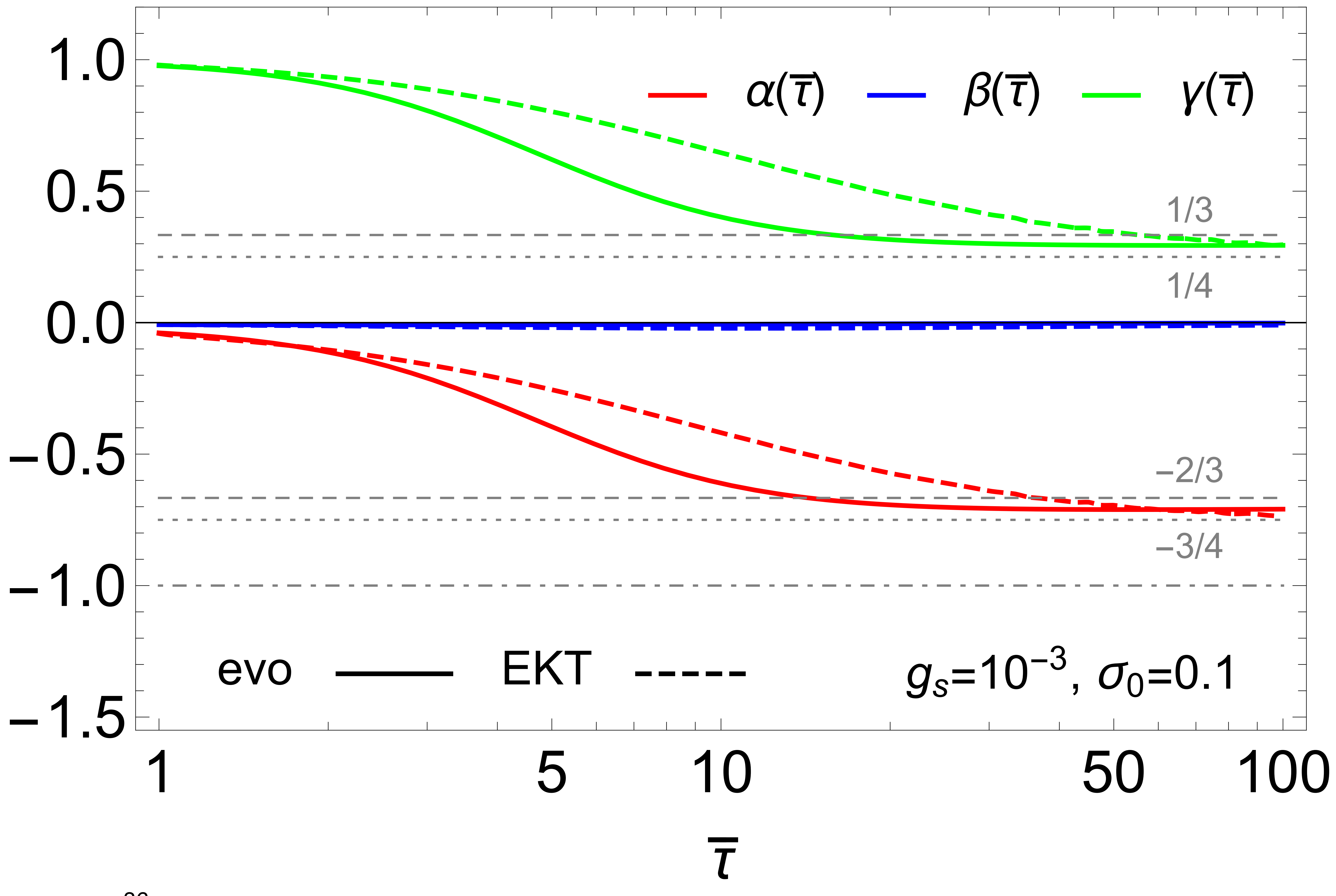
Recapitulation: Results of the previous section

low-lying energy states

- Recall that the eigenvalues of $\mathscr{H}$ in the early time regime are $\mathscr{E}_{n,m} = 2n(1-\gamma) - 2m\beta$, for $n, m = 0, 1, 2, \ldots$.

- But, $\beta \to 0$ on the BMSS fixed point (late times on the plot on the right).

$\Longrightarrow$ No substantial memory loss for the $p_\perp$ dependence of f.



Breakdown of the scaling regime

a necessary stage in the hydrodynamization process

- In the previous discussion, we showed that a distribution function f of the form

$$f = A(y)\, w\left(\frac{p_\perp}{B(y)}, \frac{p_z}{C(y)}\right), \text{with } w(\zeta, \xi) = \exp[-(\zeta^2 + \xi^2)/2]$$

is the instantaneous ground state that explains an initial stage of memory loss.

- However, hydrodynamics corresponds to

$$f = w\left(\frac{p}{T(y)}\right), \text{with } w(\chi) = [\exp(\chi) - s]^{-1},\ s \in \{-1,0,1\},$$

where the different values of s correspond to fermions, classical particles, and bosons, respectively.

Breaking the scaling regime

restoring terms in the collision kernel

- To make the approach to hydrodynamics possible, we need to restore the terms we dropped:

$$\partial_\tau f - \frac{p_z}{\tau}\partial_{p_z} f = 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f]\,\nabla_{\mathbf{p}}^2 f$$

⬇

$$\partial_\tau f - \frac{p_z}{\tau}\partial_{p_z} f = 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f]\left[I_a[f]\,\nabla_{\mathbf{p}}^2 f + I_b[f]\,\nabla_{\mathbf{p}}\cdot\left(\hat{p}(1+f)f\right)\right]$$

- We will neglect the explicit Bose enhancement in the last term in what follows. The equilibrium distribution will thus be Boltzmann instead of Bose-Einstein.

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Adiabaticity beyond scaling

how to choose a frame with adiabatic ground state evolution

- The description in the previous discussion may be cast as an expansion

$$f(\zeta B(y), \xi C(y), y) = \sum_{i,j} c_{ij}(y)\, P_{ij}(\zeta, \xi)\ \exp\{-\left(\xi^2 + \zeta^2\right)/2\},$$

where P_{ij} is a polynomial of degree i in ζ and j in ξ. This, by construction, is well-adapted to describe the ground state at early times.

- To accommodate the transition to a hydrodynamic state, we write

$$f(\chi D(y), u, y) = \sum_{n,l} c_{nl}(y)\, P_{nl}(\chi, u; r(y))\ \exp\{-\left(u^2 r^2(y)/2 + \chi\right)\},$$

where we introduced a new time-dependent variable $r(y)$ and $u \equiv p_z/p = \cos\theta$.

Scaling exponents in the new basis

- We plot

$$\beta_{p_T^2} = -(1/2)\partial_y \log\langle p_\perp^2\rangle\,,$$

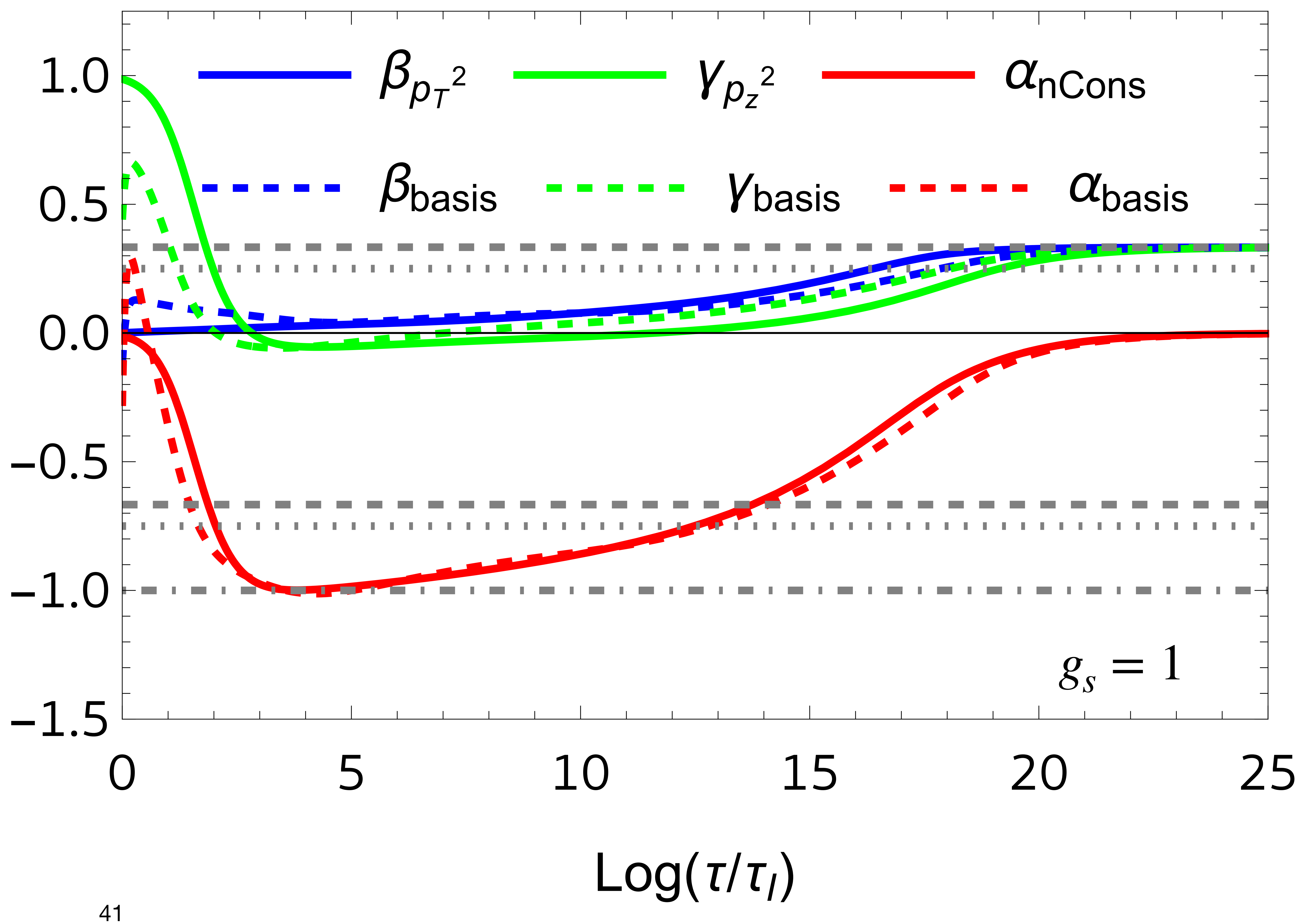
$$\gamma_{p_z^2} = -(1/2)\partial_y \log\langle p_z^2\rangle\,,$$

$$\alpha_{\rm nCons} = \gamma_{p_z^2} + 2\beta_{p_T^2} - 1\,,$$

from the solution to the kinetic equation, and also from the first basis state $\beta_{\rm basis}, \gamma_{\rm basis}, \alpha_{\rm basis}$.

- At early times (up to $\log(\tau/\tau_I) \sim 10$) we see the dilute fixed point.

- At late times, hydrodynamics.

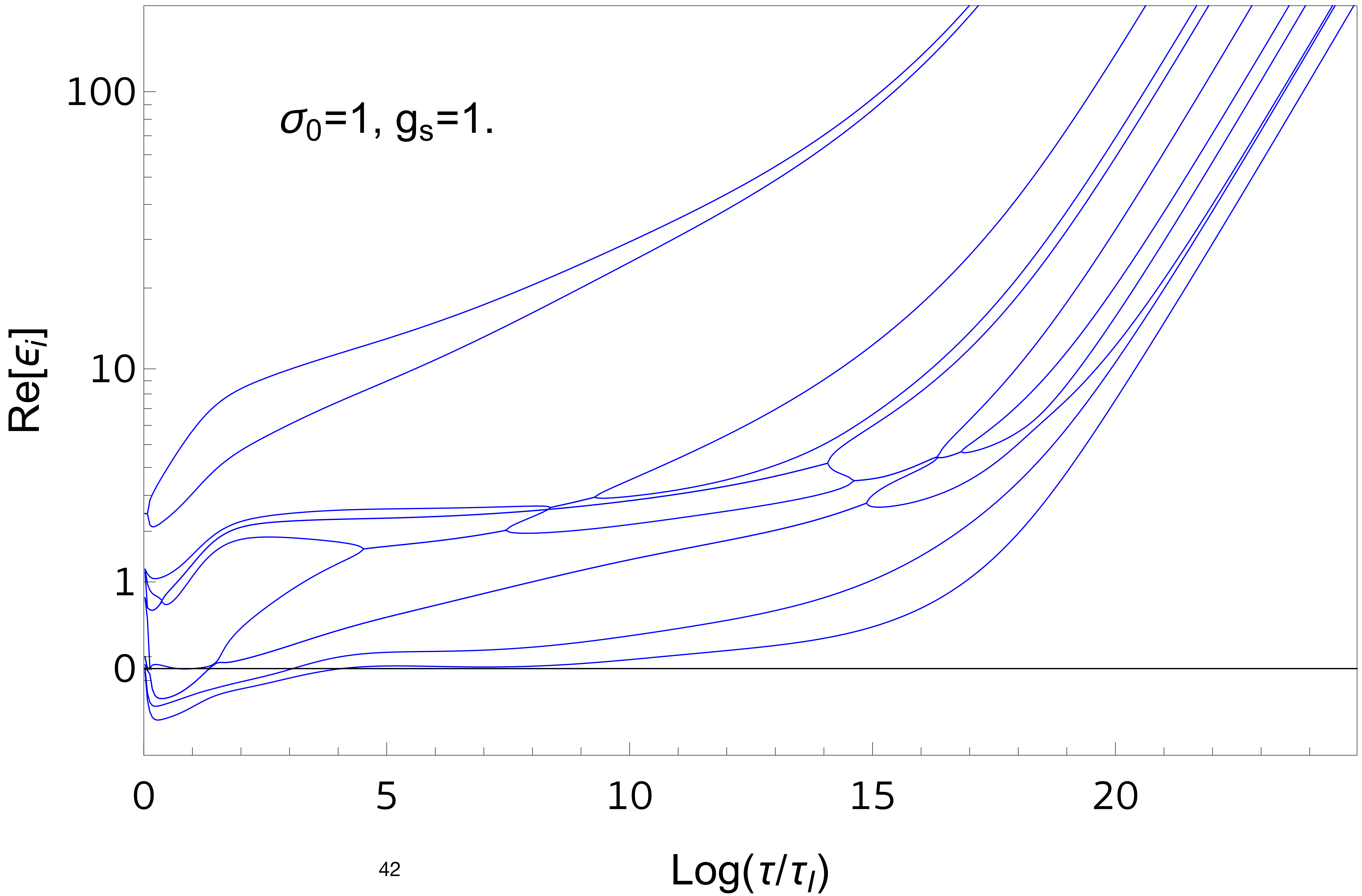


Energy levels

from early times to late times

$$f(\mathbf{p}, \tau = \tau_I) = \frac{\sigma_0}{g_s^2} e^{-\sqrt{2}p/Q_s} e^{-r_i^2 u^2/2} Q_0(u; r)$$

- We see that up until $\log(\tau/\tau_I) \sim 10$, the ground state is approximately degenerate.
- When the system approaches hydrodynamics, a gap opens and a unique ground state remains.



Eigenstate coefficients

$$f(\mathbf{p}, \tau = \tau_I) = \frac{\sigma_0}{g_s^2} e^{-\sqrt{2}p/Q_s} e^{-r_i^2 u^2/2} Q_0(u; r)$$

from early times to late times

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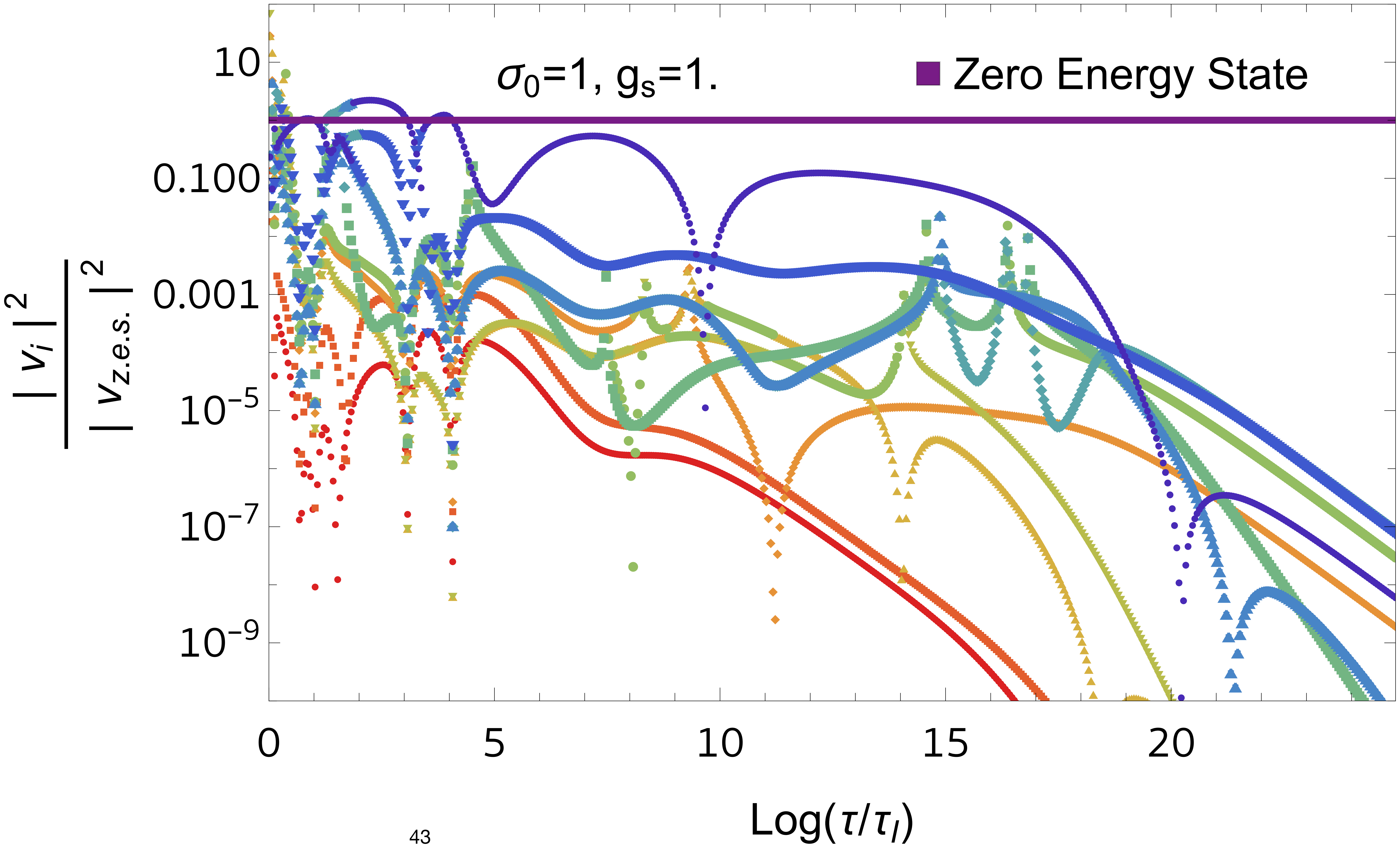

Conclusions from this study

new insights into the process of hydrodynamization

- We have shown, in a simplified version of QCD kinetic theory, that:
  - Memory of the initial condition is lost sequentially due to the opening of energy gaps that make the information in excited states decay quickly.
  - In each scaling regime, the ground state(s) evolve adiabatically, either by themselves or as a set, and high-energy modes effectively decouple from the dynamics.
- Future work:
  - Include $1 \leftrightarrow 2$ processes in the collision kernel, so as to be able to apply the AH framework in a setting where hydrodynamization is rapid, as in HICs.
  - Include a nontrivial profile in position space, emulating the fireball formed in a HIC.

Extra slides

‘Optimizing’ adiabaticity

rescaling the degrees of freedom

- From the previous discussion, we see that scaling plays a crucial role in this problem.

- This gives us a very useful tool to ‘optimize’ adiabaticity. For instance, if we have a distribution function evolving as

$$f(p_\perp, p_z, \tau) = A(\tau)\, w\Big(p_\perp/B(\tau), p_z/C(\tau); \tau\Big),$$

then we can look for the choice of A, B, C that maximize the degree to which the dynamics of w is adiabatic.

- We take $|\psi\rangle \leftrightarrow w(\zeta, \xi; \tau)$.

$q[f;\tau] = 4\pi\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f]\tau$

'Optimizing' adiabaticity

in practice

- The original kinetic equation has the form

$$\tau\partial_\tau f - p_z\partial_{p_z} f = q[f;\tau]\,\nabla^2_{\mathbf{p}} f\,.$$

- This is a linear equation of motion, except for the non-linear dependence through $q[f;\tau]$.

- Nothing prevents us from making the replacement $q[f;\tau] \to q(\tau)$, solve the equation for an arbitrary $q(\tau)$, and in the end replace the resulting distribution $f[q(\tau)]$ in the definition of q and solve self-consistently:

$$q(\tau) = q[f[q(\tau)];\tau].$$

‘Optimizing’ adiabaticity
in practice

- One can then write the kinetic equation for w as

$$\partial_y w = -\mathscr{H} w \,,$$

with $$\mathscr{H} = \alpha - (1-\gamma)\Big[\tilde{q}\,\partial_\xi^2 + \xi\,\partial_\xi\Big] + \beta\,\Big[\tilde{q}_B\,(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta) + \zeta\,\partial_\zeta\Big].$$

For brevity, we have denoted

$$\tilde{q} = \frac{q}{C^2(1-\gamma)}\,, \quad \tilde{q}_B \equiv -\frac{q}{B^2\beta}\,.$$

What is the advantage of this?

- Because A, B, C are a choice of coordinates (a “gauge” choice to describe the system), we can choose them such that $\tilde{q} = \tilde{q}_B = 1$.

How?

Note that

$$\tilde{q}(\tau) = \frac{q(\tau)}{C^2(\tau)(1-\gamma(\tau))} \implies \gamma(\tau) = \boxed{-\frac{\tau\partial_\tau C}{C} = 1 - \frac{q(\tau)}{\tilde{q}(\tau)C^2},}$$

Differential equation for $C(\tau)$

$\implies$ we can choose $\tilde{q}$ by “fixing the gauge” and choosing $C(\tau)$.

$\tilde{q} = 1$ corresponds to fixing $C(\tau)$ by solving: $-\dfrac{\tau\partial_\tau C}{C} = 1 - \dfrac{q(\tau)}{C^2}$. Same for β and $\tilde{q}_B$.

Results

low-lying energy states

- We can choose A such that $\alpha = \gamma + 2\beta - 1$ to set the ground state energy $\mathscr{E}_{0,0} = 0$.
- The eigenvalues of $\mathscr{H}$ are $\mathscr{E}_{n,m} = 2n(1-\gamma) - 2m\beta \ , \quad n, m = 0, 1, 2, \ \dots$
- The left and right eigenstates are:

$$\phi^L_{n,m} = \mathrm{He}_{2n}\big(\xi\big)\ {}_1F_1\left(-2m, 1, \frac{\zeta^2}{2}\right),$$

$$\phi^R_{n,m} = \frac{1}{\sqrt{2\pi}\,(2n)!}\mathrm{He}_{2n}\big(\xi\big)\ {}_1F_1\left(-2m, 1, \frac{\zeta^2}{2}\right)\exp\left(-\frac{\xi^2}{2} - \frac{\zeta^2}{2}\right)$$

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Gapped energy levels!
$\Longrightarrow$ Ground state will dominate after a transient time

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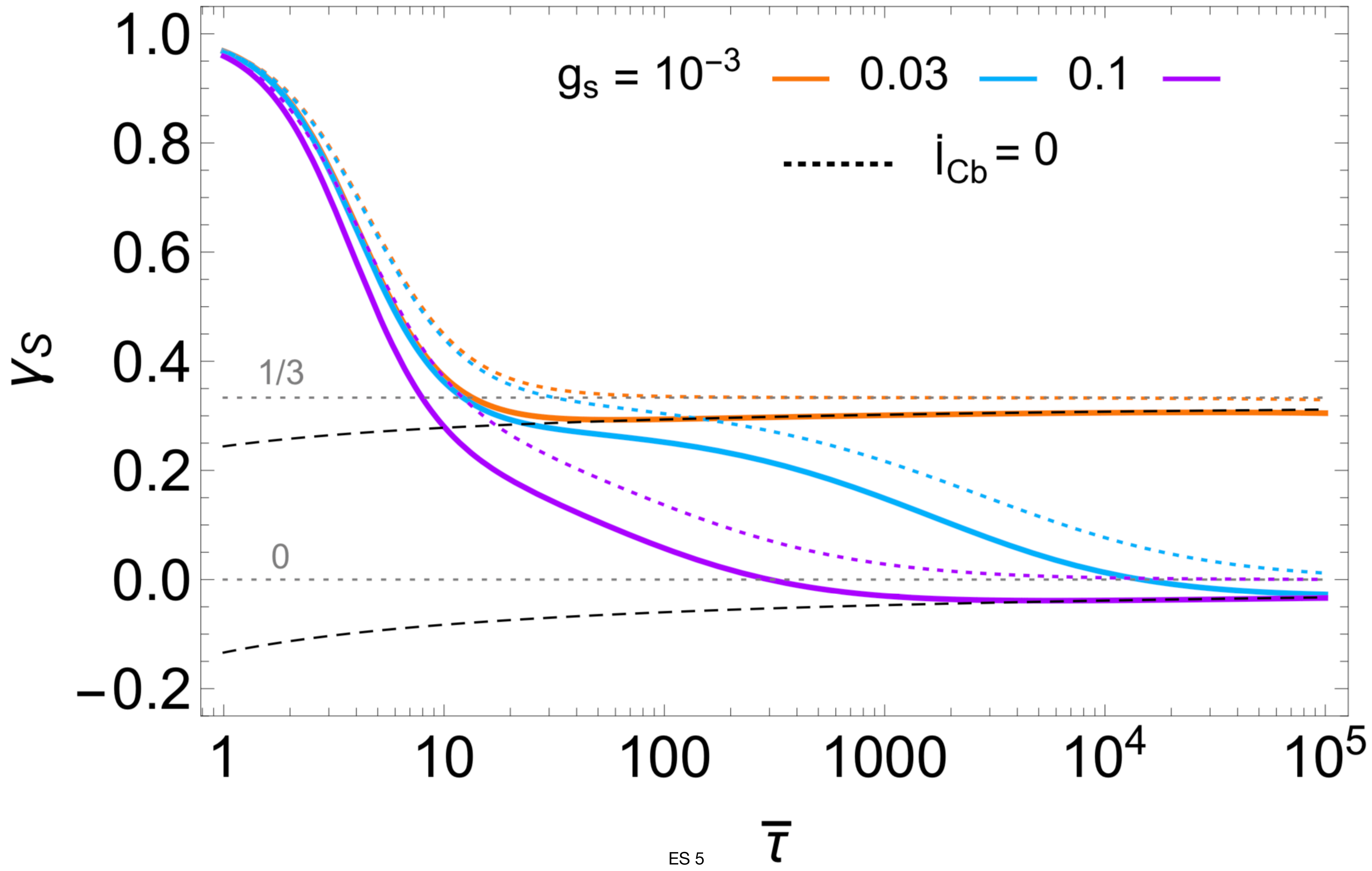
Left and right eigenstates differ because $\mathscr{H}$ is not hermitian

Evolution of the exponents for different coupling strengths



$\sigma_0 = 0.1$

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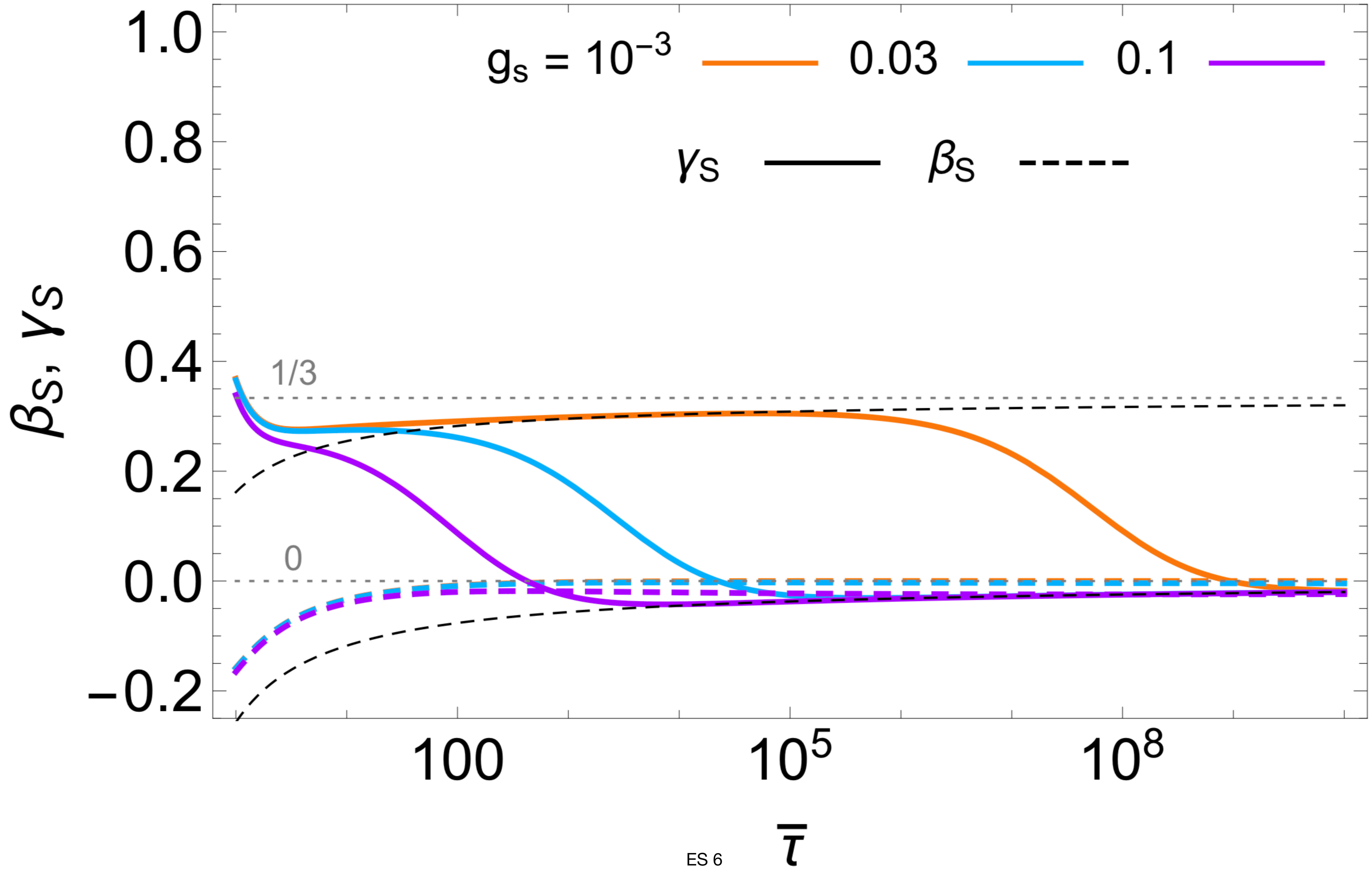


$\sigma_0 = 0.6$

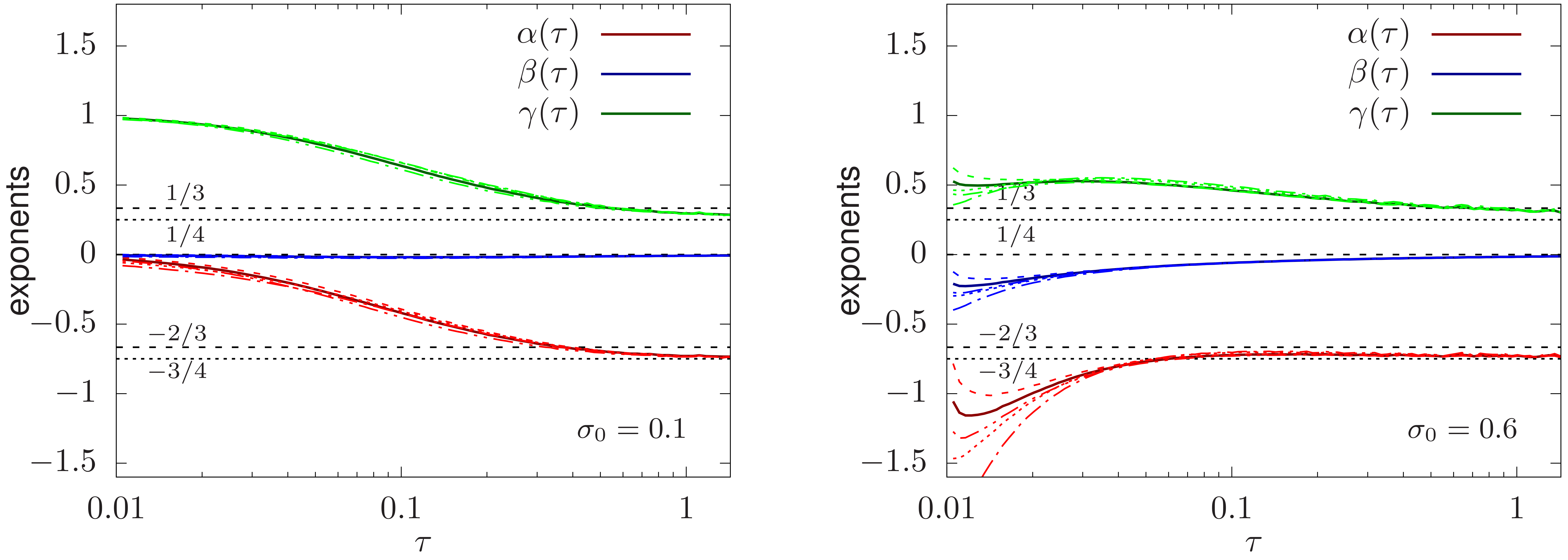
$f(\tau_I) = \frac{\sigma_0}{g_s^2} \exp\left(-\frac{p_\perp^2 + \xi^2 p_z^2}{Q_s^2}\right)$; $\xi = 2$, $Q_s\tau_I = 70$, $g_s = 10^{-3}$

Evidence for AH in QCD effective kinetic theory

by A. Mazeliauskas, J. Berges [6]

- After a transient time, [6] observed that f_g took a time-dependent scaling form

$$f(p_\perp, p_z, \tau) = e^{\int^\tau \alpha(\tau')\,\mathrm{d}\ln\tau'} f_S\left(e^{\int^\tau \beta(\tau')\,\mathrm{d}\ln\tau'} p_\perp, e^{\int^\tau \gamma(\tau')\,\mathrm{d}\ln\tau'} p_z\right).$$



[6] A. Mazeliauskas, J. Berges, “Prescaling and far-from-equilibrium hydrodynamics in the quark-gluon plasma” Phys. Rev. Lett. 122, 122301 (2019)

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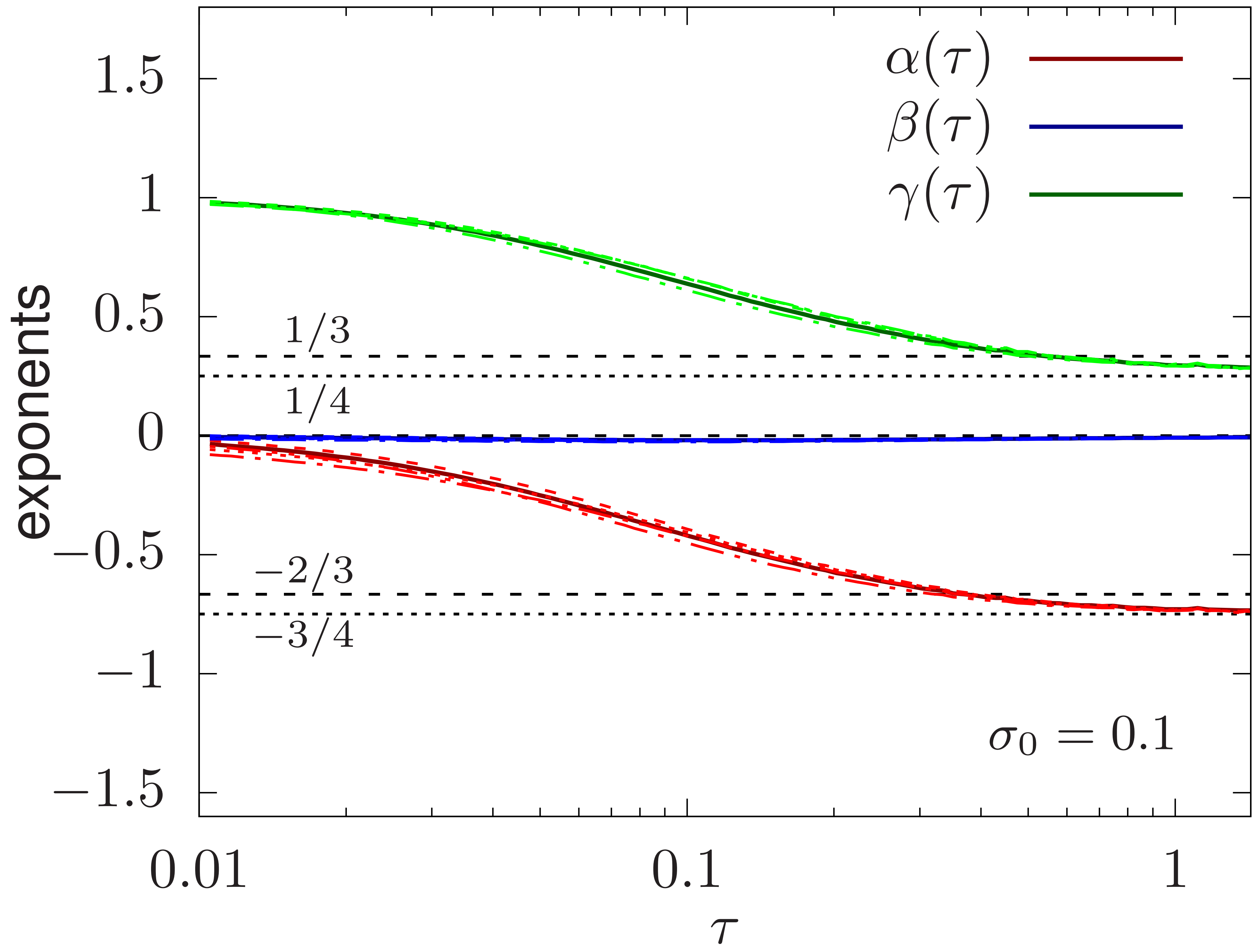

In the plots, the exponents were obtained by taking moments of the distribution function:

$$\mathrm{n}_{m,n}(\tau) = \int_{\mathbf{p}} p_\perp^m |p_z|^n f(p_\perp, p_z; \tau),$$

and using that, if scaling takes place,

$$\frac{\partial_\tau \ln \mathrm{n}_{m,n}}{\partial \ln \tau} = \alpha(\tau) - (m+2)\,\beta(\tau) - (n+1)\,\gamma(\tau)$$

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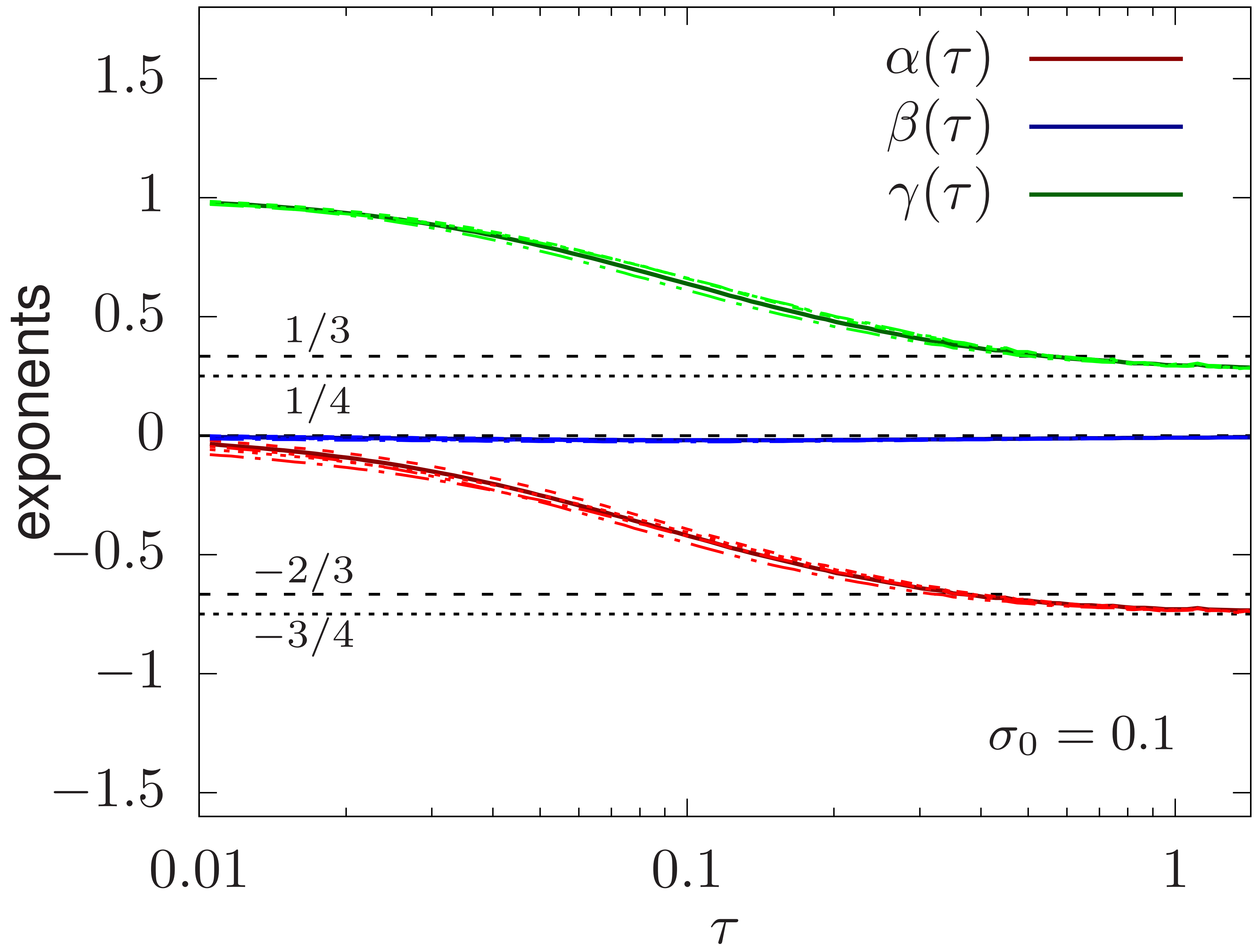

Then, one can use triads of moments to obtain α, β, γ. For example, if we use $\mathrm{n}_{0,0}, \mathrm{n}_{1,0}, \mathrm{n}_{0,1}$,

$$\alpha = 4\partial_{\ln\tau} \ln \mathrm{n}_{0,0} - 2\partial_{\ln\tau} \ln \mathrm{n}_{1,0} - \partial_{\ln\tau} \ln \mathrm{n}_{0,1}\,,$$

$$\beta = \partial_{\ln\tau} \ln \mathrm{n}_{0,0} - \partial_{\ln\tau} \ln \mathrm{n}_{1,0}\,,$$

$$\gamma = \partial_{\ln\tau} \ln \mathrm{n}_{0,0} - \partial_{\ln\tau} \ln \mathrm{n}_{0,1}\,.$$

If every triad of moments gives the same α, β, γ, then the distribution has the above scaling form.

Curves in the figure $\Longleftrightarrow$ different triad choices.

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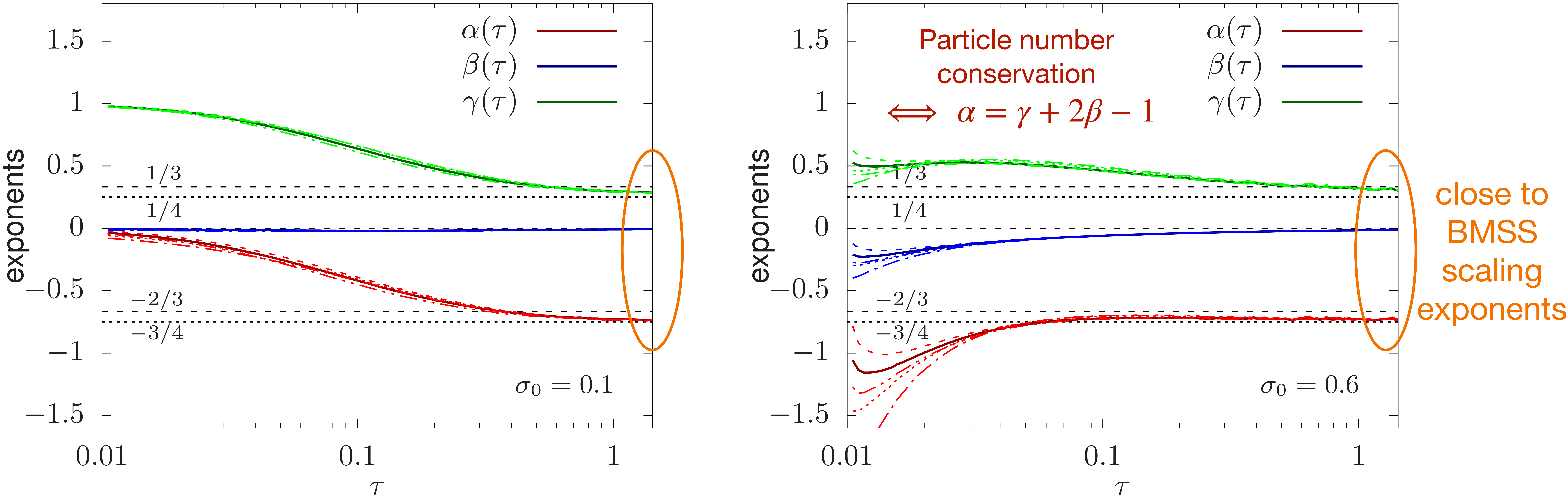

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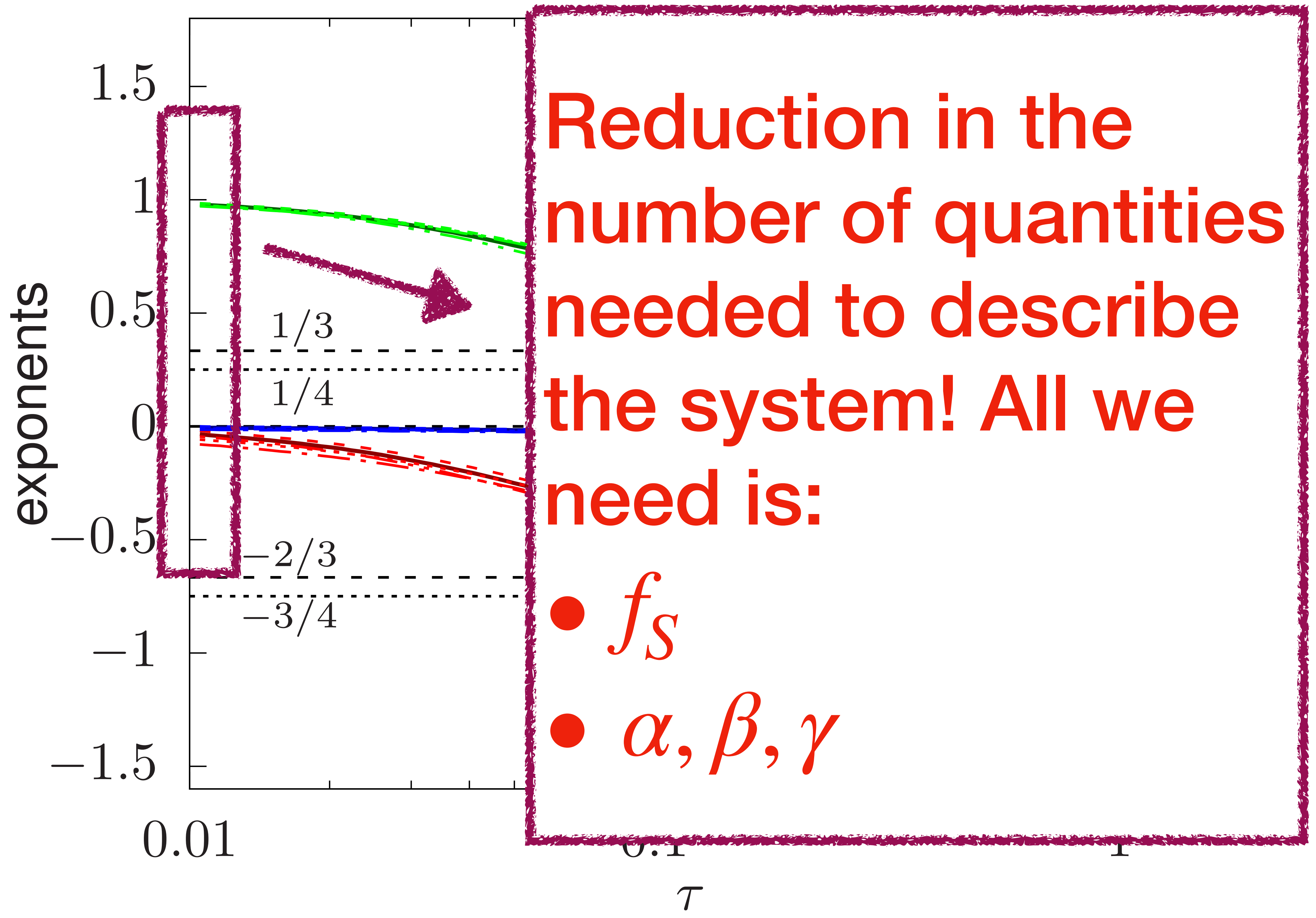
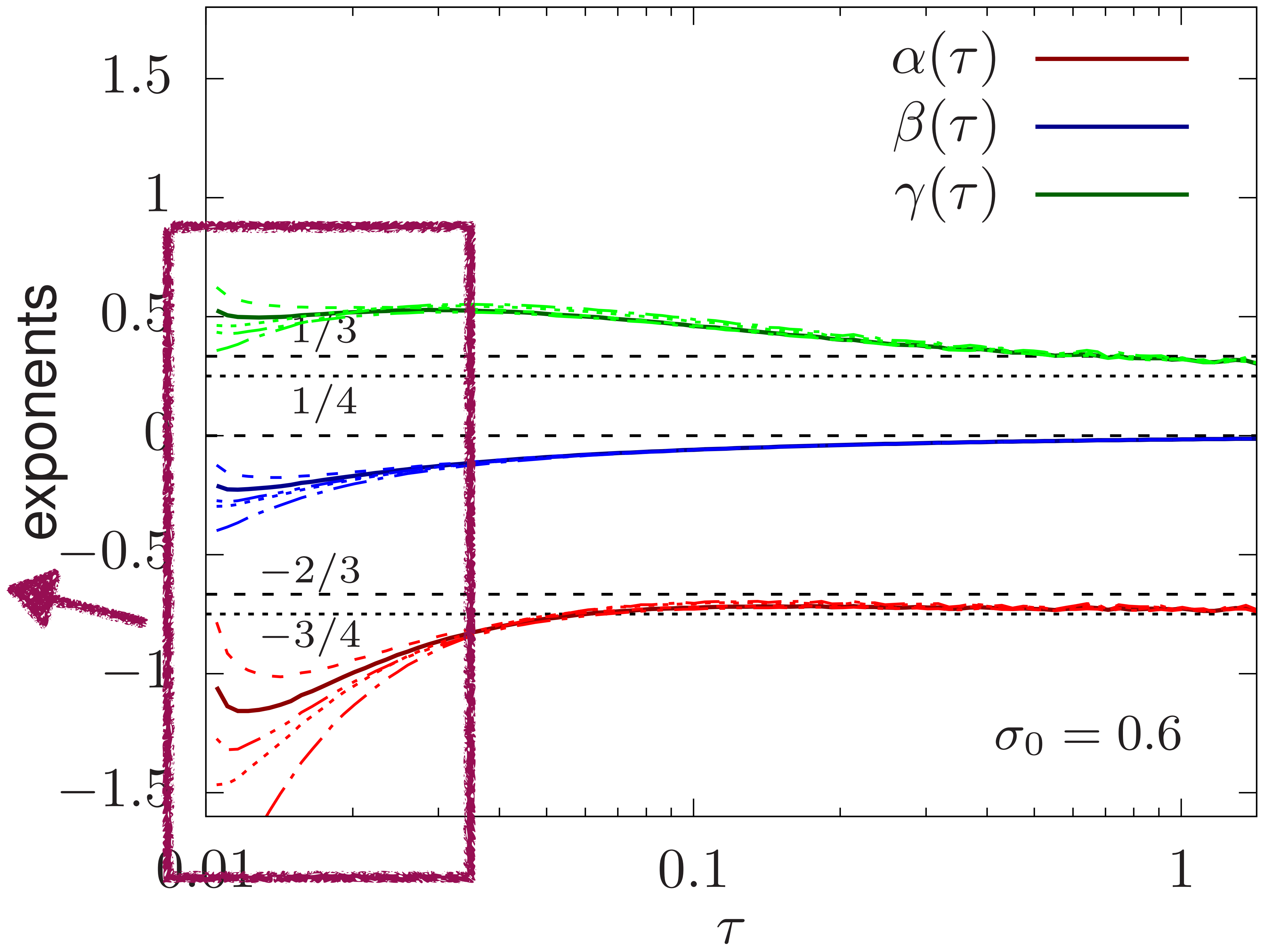


Reduction in the number of quantities needed to describe the system! All we need is:

- f_S
- α, β, γ



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