Efficient Parallel Numerical Simulations of the Einstein Equations in Spherical Coordinates

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INT Program 23-2

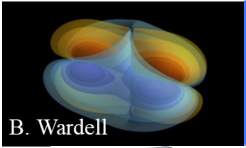
RIT | College of Science
Center for Computational Relativity and Gravitation

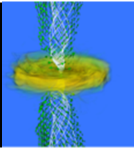
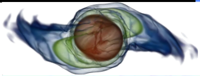
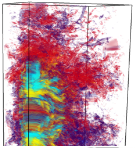
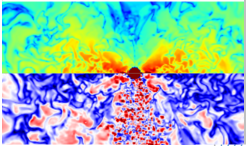
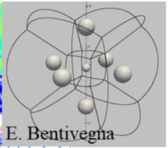
Einstein Toolkit

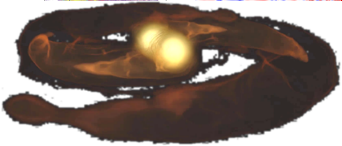
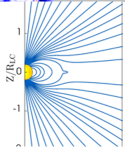
- Collection of scientific software components and tools to simulate and analyze General Relativistic Astrophysical systems
- Freely available as open source at http://www.einsteintoolkit.org
- State-of-the-art set of tools for numerical relativity, open source
- Currently 402 members from 282 sites and 49 countries
- $>$ 428 publications, $>$ 57 theses building on these components (as of June 2023)
- Regular, tested releases
- User support through various channels

Science

- Binary Black Hole Mergers
- Neutron Star Mergers
- Supernovae
- Accretion Disks
- Boson Stars
- Hairy Black Holes
- Cosmic Censorship

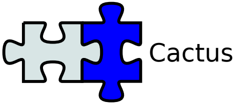


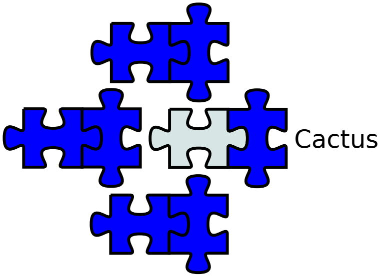
Einstein Toolkit as growing project

- Initially: some infrastructure, some application code



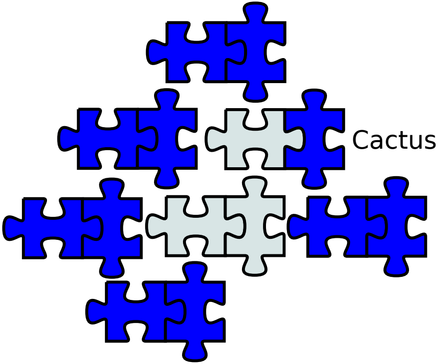
Einstein Toolkit as growing project

- Growing application suite



Einstein Toolkit as growing project

- Growing infrastructure "return"



Numerical Relativity in Spherical Coordinates

- Spherical coordinates vs Cartesian coordinates.
  - Take advantage of approximate symmetries in a number of astrophysical objects (single stars, black holes, accretion disks, post merger remnant)
  - Lower numerical dissipation in the evolution of fluid angular momentum.

SphericalNR [Mewes++ PRD 2018, Mewes++ PRD 2020]

- Implementaion in the Einstein Toolkit
  - Identify the (x, y, z) in Carpet with (r, θ, ϕ)
  - Apply internal boundary condition with SphericalBC

$r = 0$	$\theta = 0, \pi$	$\phi = 0, 2\pi$
Parity	Parity	Periodicity

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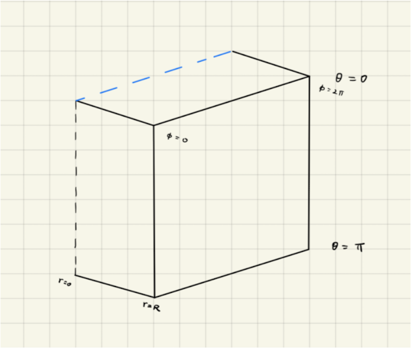

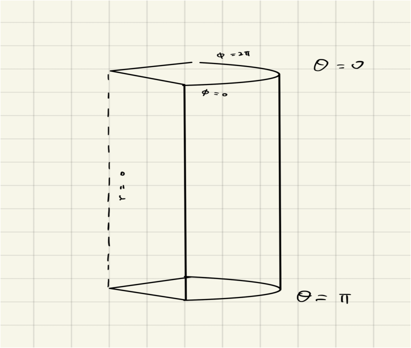

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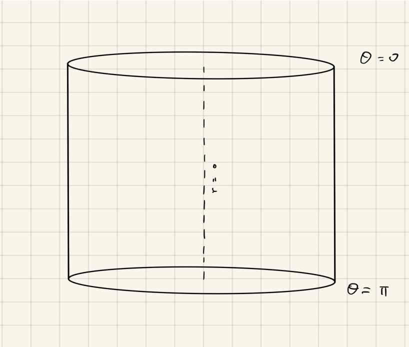
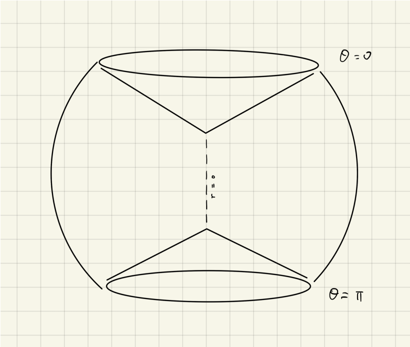
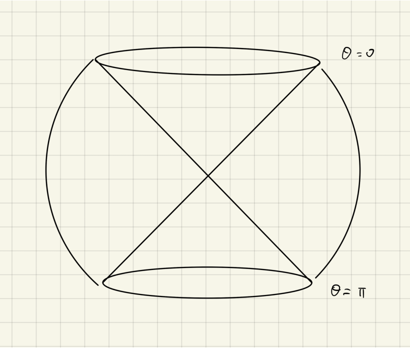
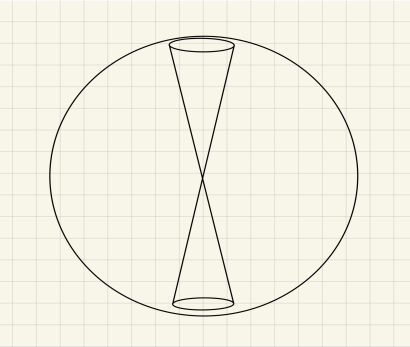




- Implementaion in the EINSTEIN TOOLKIT
  - Identify the (x, y, z) in CARPET with (r, θ, ϕ)
  - Apply internal boundary condition with SPHERICALBC

$r = 0$	$\theta = 0, \pi$	$\phi = 0, 2\pi$
Parity	Parity	Periodicity





- Coordinate singularities $\rightarrow$ handled analytically
  [Baumgarte++ PRD 2013, Baumgarte++ PRD 2015, Ruchlin++ PRD 2018]
  - Reference-metric version of BSSN/fCCZ4 formulation and GRMHD
  - Proper rescaling of tensorial quantities
- Reference Metric Formulation

$$
\begin{aligned}
h_{ij} &= \hat{\mathbf{e}}_i^{\{l\}} \hat{\mathbf{e}}_j^{\{m\}} h_{\{k\}\{l\}}, \\
\partial_k h_{ij} &= \hat{\mathbf{e}}_i^{\{l\}} \hat{\mathbf{e}}_j^{\{m\}} \partial_k h_{\{l\}\{m\}} + h_{\{l\}\{m\}} \partial_k \left(\hat{\mathbf{e}}_i^{\{l\}} \hat{\mathbf{e}}_j^{\{m\}} \right)
\end{aligned}
$$

- GRMHD

$$
\begin{array}{lcl}
\partial_t D + \partial_k \left(f_D\right)^k & = & -\hat{\Gamma}^k{}_{kl} \left(f_D\right)^l , \\
\partial_t S_i + \partial_k \left(f_S\right)_i{}^k & = & \left(s_S\right)_i - \hat{\Gamma}^k{}_{kl} \left(f_S\right)_i{}^l + \hat{\Gamma}^l{}_{ki} \left(f_S\right)_l{}^k , \\
\partial_t \tau + \partial_k \left(f_\tau\right)^k & = & s_\tau - \hat{\Gamma}^k{}_{kl} \left(f_\tau\right)^l , \\
\partial_t A_i & = & \alpha \hat{\epsilon}_{ijk} \bar{v}^j \mathcal{B}^k - \partial_i \left(\alpha \Phi - \beta^k A_k\right) , \\
\partial_t \hat{\Phi} + \partial_k \left(f_\Phi\right)^k & = & -\zeta \alpha \hat{\Phi} - \hat{\Gamma}^l{}_{kl} \left(f_\Phi\right)^l , \\
\mathcal{B}^i & = & \hat{\epsilon}^{ijk} \partial_j A_k .
\end{array}
$$

- GRMHD in Reference Metric Formulation

$$
\begin{array}{lcl}
\partial_t D + \sigma^m_{\{k\}\{l\}} \hat{\mathcal{R}}^{\{k\}} \partial_m \left(f_D\right)^{\{l\}} & = & \Omega_D , \\
\partial_t S_{\{i\}} + \sigma^m_{\{k\}\{l\}} \hat{\mathcal{R}}^{\{k\}} \partial_m \left(f_S\right)_{\{i\}}{}^{\{l\}} & = & \left(\Omega_S\right)_{\{i\}} , \\
\partial_t \tau + \sigma^m_{\{k\}\{l\}} \hat{\mathcal{R}}^{\{k\}} \partial_m \left(f_\tau\right)^{\{l\}} & = & \left(\Omega_\tau\right) , \\
\partial_t A_{\{i\}} & = & \left(\Omega_A\right)_{\{i\}} , \\
\partial_t \hat{\Phi} + \sigma^m_{\{k\}\{l\}} \hat{\mathcal{R}}^{\{k\}} \partial_m \left(f_\Phi\right)^{\{k\}} & = & \Omega_\Phi , \\
\mathcal{B}^{\{i\}} & = & \sigma_l^{\{i\}\{m\}} \hat{\mathcal{R}}_{\{m\}} \hat{\epsilon}^{ljk} \partial_j A_k .
\end{array}
$$

Filtering

- Severe CFL time step restriction

$$dt = \mathcal{C} \min \left[dr, \frac{dr}{2} d\theta, \frac{dr}{2} \sin\left(\frac{d\theta}{2}\right) d\phi \right]$$

- Dual-FFT filtering (damping functions suppress CFL unstable modes)

Filtering

- Double covering $\theta \in [0, \pi] \to \vartheta \in [0, 2\pi]$

$$\mathbf{X}(r, \vartheta, \phi) = \begin{cases} \mathbf{X}(r, \theta, \phi), & \vartheta \in [0, \pi] \\ (-1)^a \mathbf{X}(r, 2\pi - \theta, \pi + \phi), & \vartheta \in [\pi, 2\pi] \end{cases}$$

- Filtering θ first

$$\mathbf{X}(r, \vartheta, \phi) \xrightarrow{\text{FFT}} \tilde{\mathbf{X}}(r, l, \phi) \to f(l, l_{\max})\tilde{\mathbf{X}}(r, l, \phi) \xrightarrow{\text{iFFT}} \mathbf{X}(r, \vartheta, \phi)$$

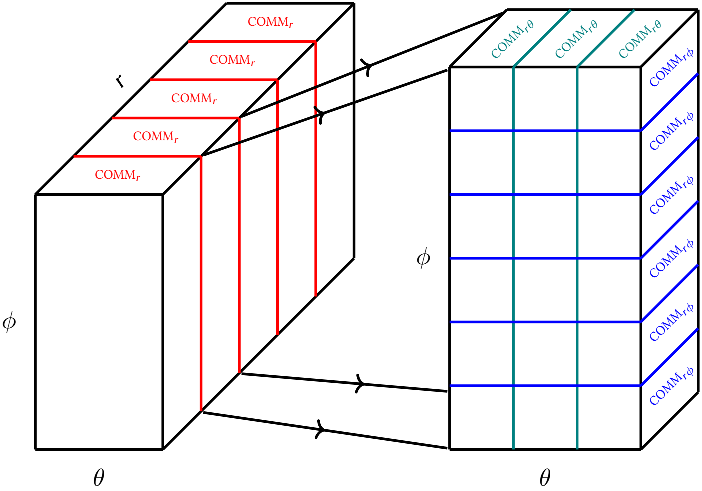
where $l_{\mathsf{max}} = \max\left(2, \frac{2r}{dr}\mathcal{L}\right)$

- Filtering ϕ next

$$\mathbf{X}(r, \theta, \phi) \xrightarrow{\text{FFT}} \tilde{\mathbf{X}}(r, \theta, m) \to f(m, m_{\max})\tilde{\mathbf{X}}(r, \theta, m) \xrightarrow{\text{iFFT}} \mathbf{X}(r, \theta, \phi)$$

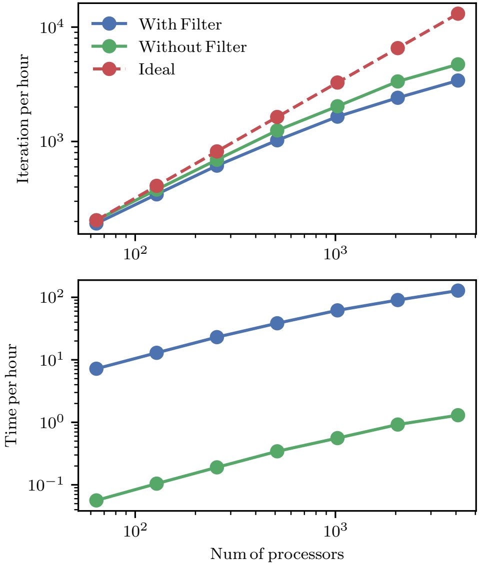
where $m_{\mathsf{max}} = \max\left(2, \frac{2r}{dr}\mathcal{L}\sin\theta\right)$

Parallelization



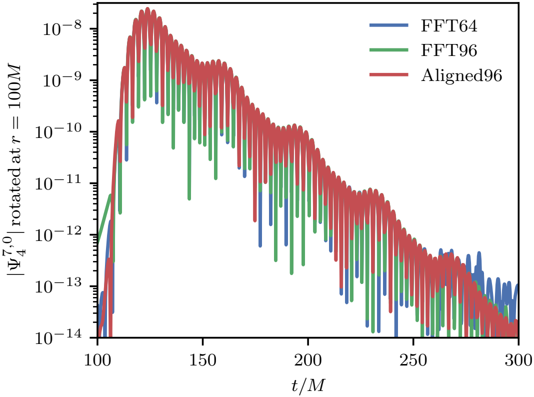
Strong Scaling Test of SphericalNR on Frontera

- Grid $n_r \times n_\theta \times n_\phi = 256 \times 128 \times 256$
- Non-filtered algorithm is 1.38 faster in iteration per hour at 4096 cores
- Filtered algorithm is roughly 100 times faster in time per hour at 4096 cores



Spinning Bowen-York BH

- Contains radiation due to the initial data being conformally flat
- Aligned with the polar *z*-axis vs *y*-axis



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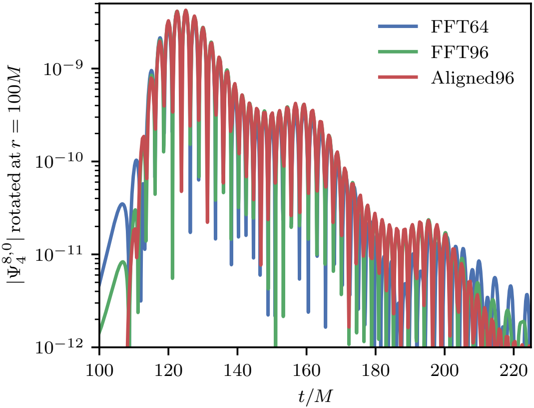
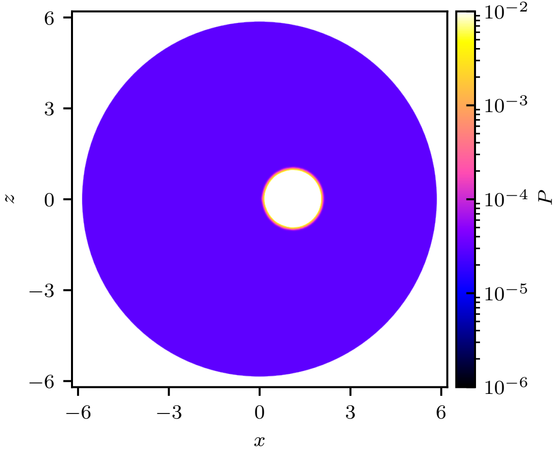


Off-Center Spherical Explosion

- Over-dense ($\rho = 1 \times 10^{-2}$, $p = 1.0$) ball of radius 1.0
- Constant magnitude magnetic field ($B^z = 0.1$), rotate by 45° about x-axis
- The center of the over-dense region has been moved to ($x = 1.1, y = z = 0$)



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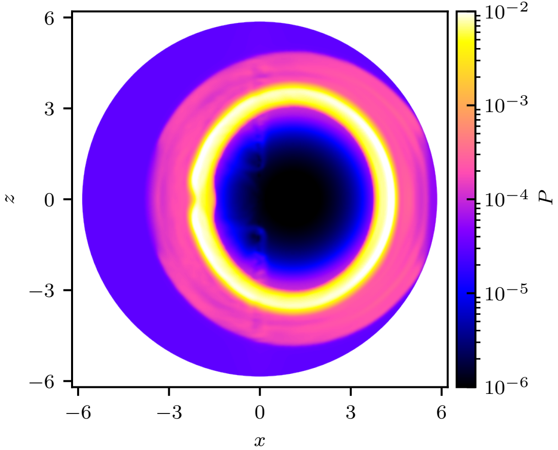

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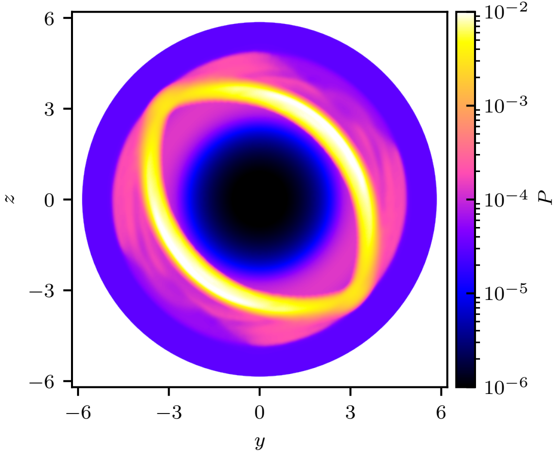
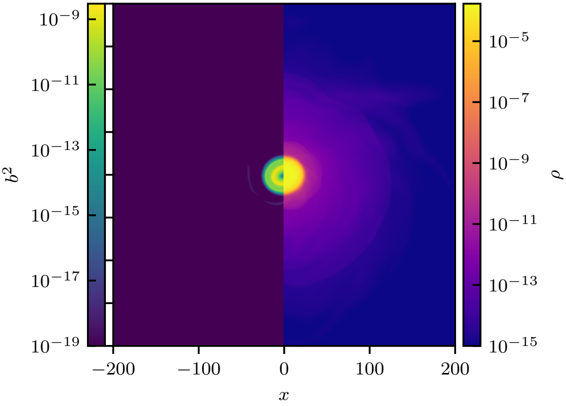


Bar-Mode Unstable Magnetized NS

- Proxy for the post merger remnant of BNS merger
- U11 model, perturb the pressure by 5% with random noise
- Initial poloidal magnetic field ($b^2 \sim 10^{-5} G$)



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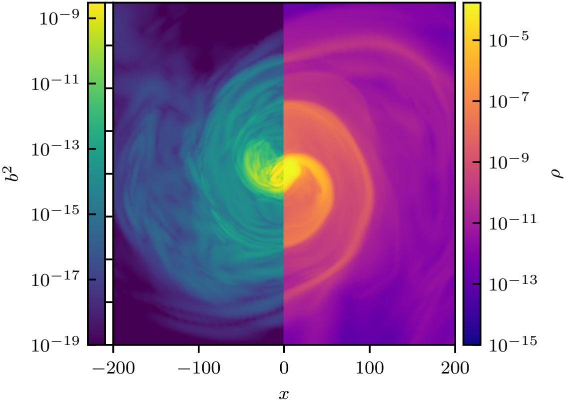

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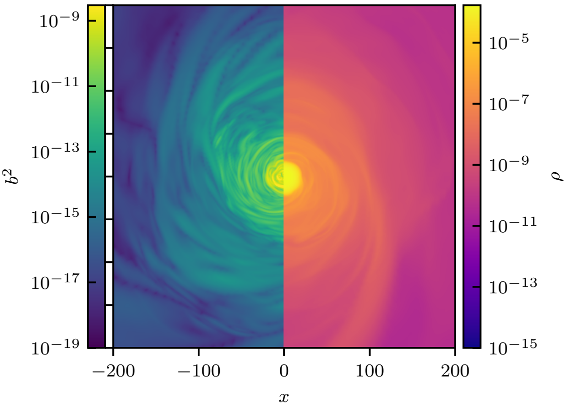

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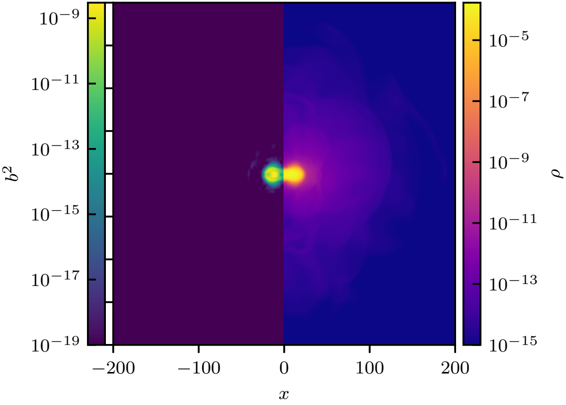

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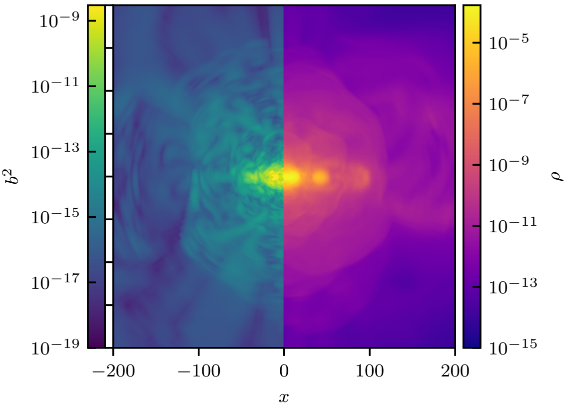

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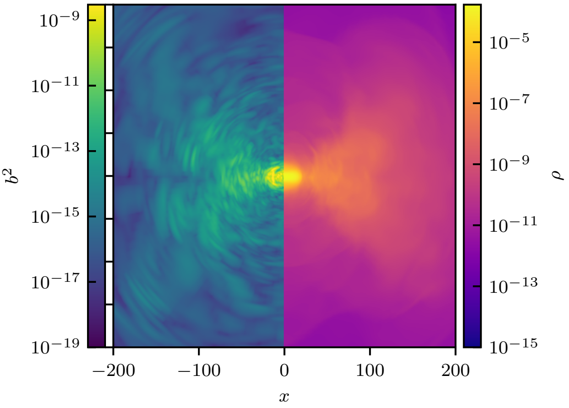

SphericalNReX

- ```
  SphThetaPhiRIndexMapping index_mapping(domain)
  ```
  ```

SphericalNReX

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  ```
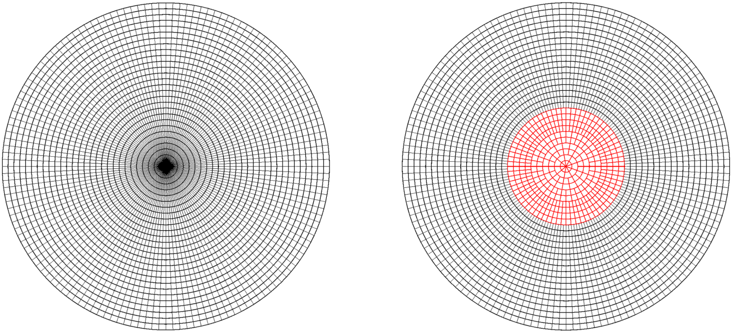
- AMR in Φ (Dang++ 2021) + Squish θ

(a) Original Polar Grid (b) Effective Grid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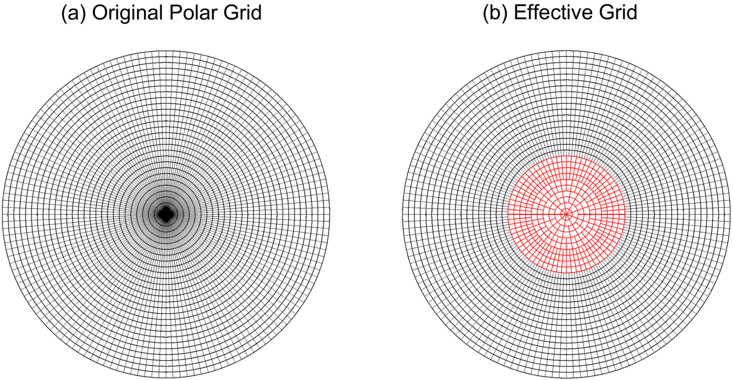

- Neutrino transport ...