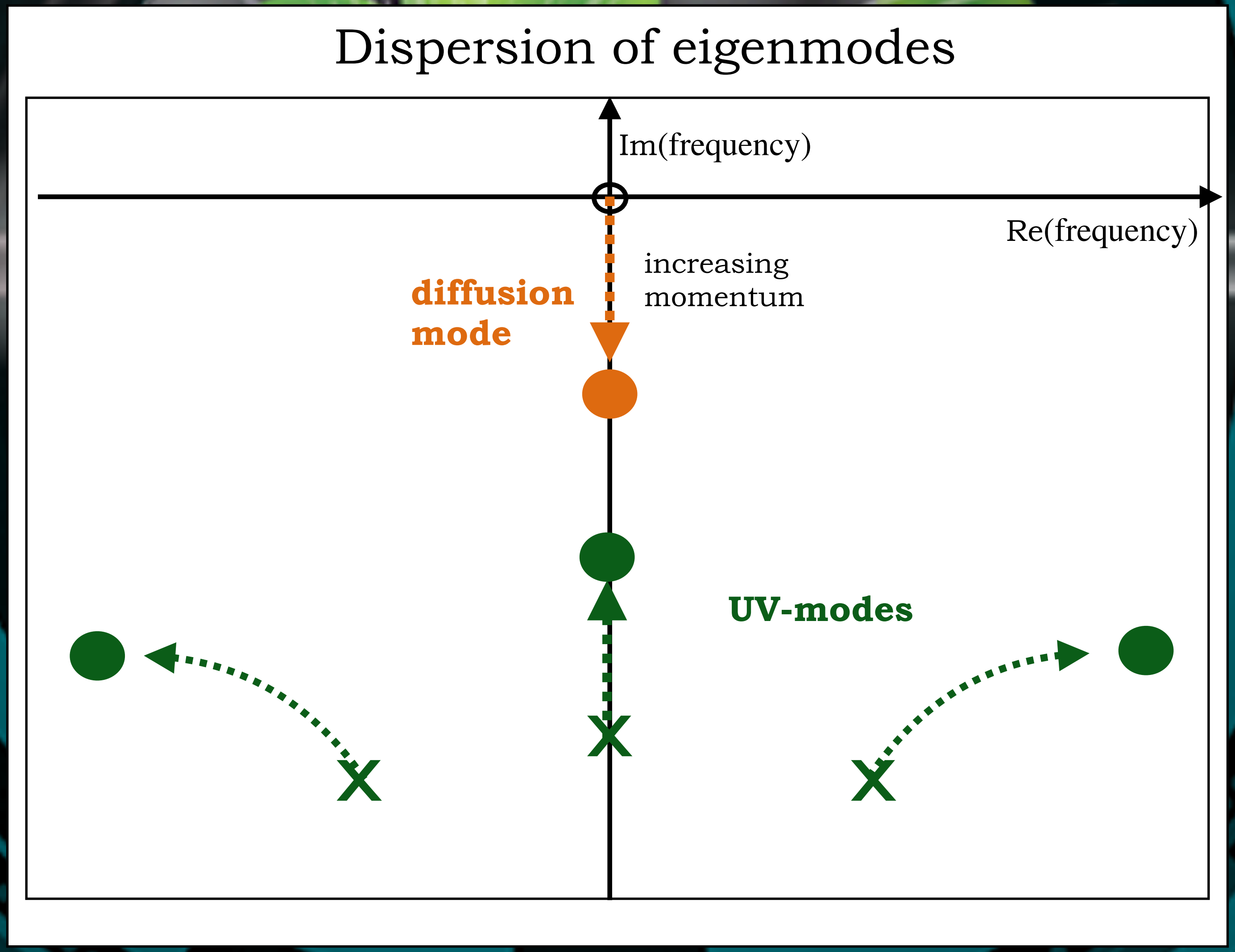
Causal and stable hydrodynamics including statistical and quantum fluctuations

Heavy Ion Physics in the EIC Era, INT, Seattle, WA

August 20th, 2024



[Abbasi, Kaminski, Tavakol; PRL (2024)]

Section 5.2 in White Paper [Sorensen et al.; Prog.Part.Nucl.Phys. (2024)]

[EIC, BNL (2024)]

Matthias Kaminski

University of Alabama

How can I be useful in the EIC Era?

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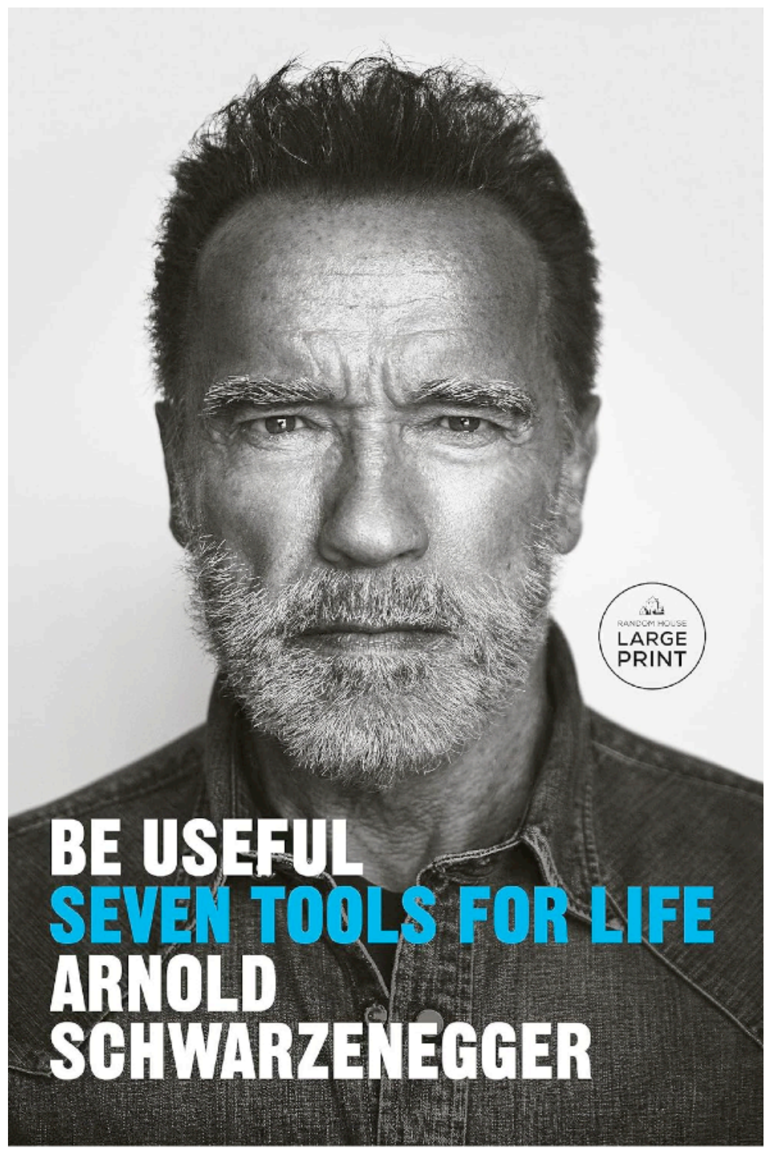
The internet's answer:

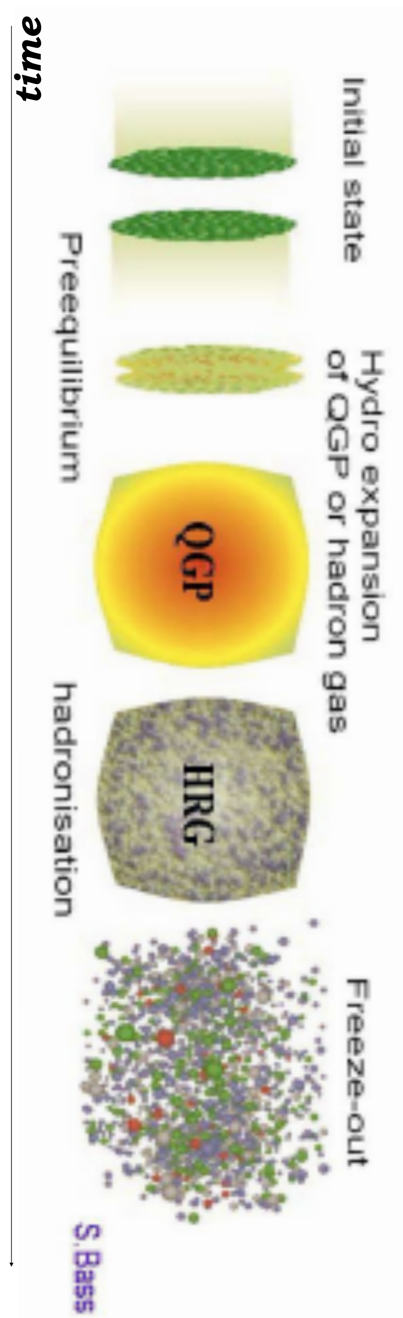


1. **Have a clear vision**
2. Never think small
3. Work your ass off
4. Sell, sell, sell
5. **Shift gears**
6. Shut your mouth, open your mind
7. **Break your mirrors**

A winning team: hydrodynamics and holography

4-page review in my Section 5.2 on Hydrodynamics
in White Paper [Sorensen et al.; Prog.Part.Nucl.Phys. (2024)]

EFFECTIVE FIELD THEORY: HYDRODYNAMICS & THERMODYNAMICS



HOLOGRAPHY (AdS/CFT)

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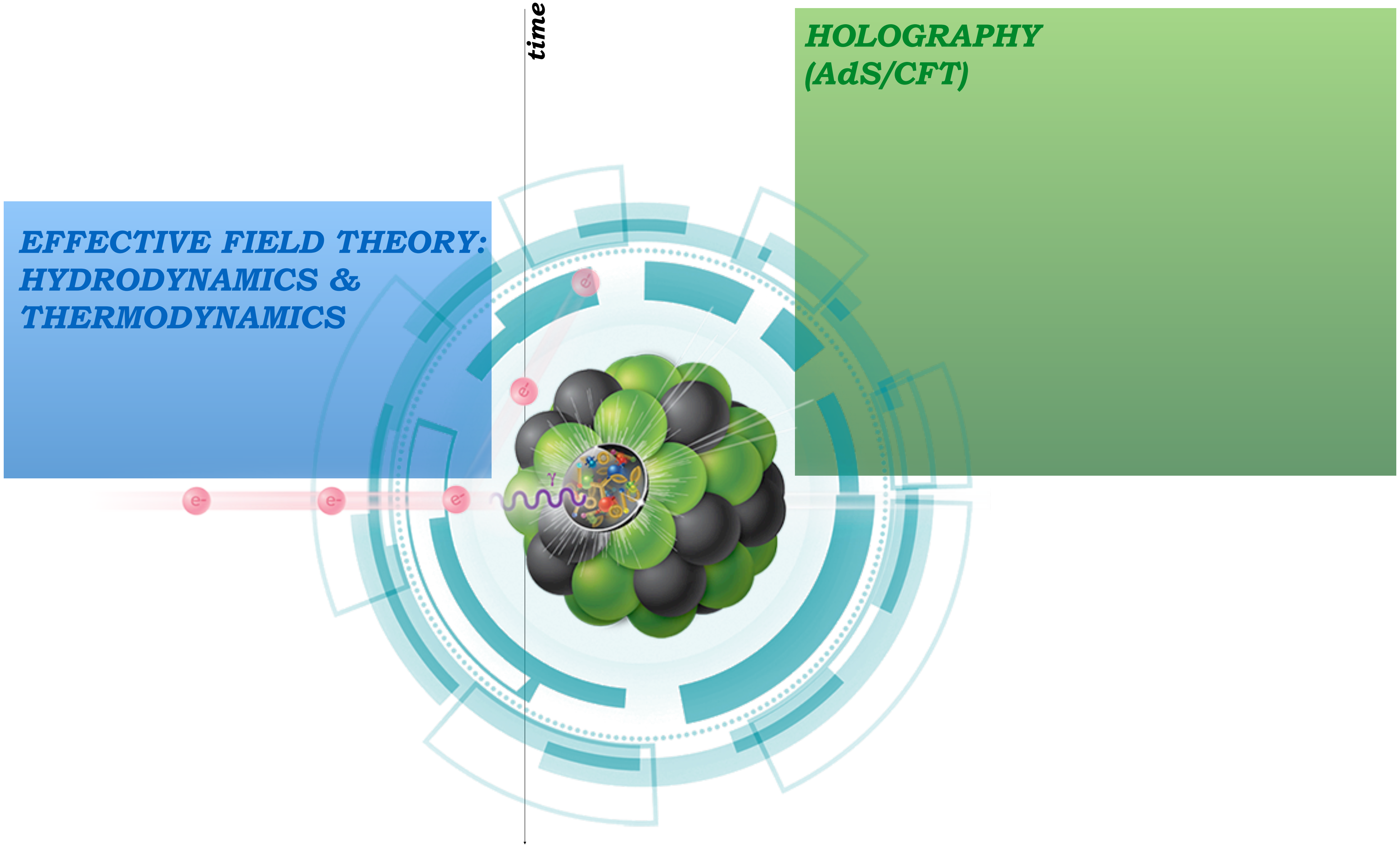

➡What can hydrodynamics and holography do for the EIC?

Hydrodynamics' Progress

Hydrodynamics involves two expansions into

1. **gradients**: zeroth order is thermodynamics, **first order includes viscosity**
2. **amplitudes**: first order is ***linearized hydrodynamics***

1960s/1970s — *1987* — *2012* — *2019* — *2023* — ***time***

Muller-Israel-Stewart (MIS): stable

textbook relativistic hydrodynamics *[Landau, Lifshitz]*

- acausal, unstable *[Hiscock & Lindblom; PRD (1985)]*
- no fluctuations (linearized)
- no generating functional (no derivation)

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Hydrodynamic modeling used to interpret data mostly: DNMR hydrodynamics
- derived from Boltzmann equation
- stable (a version of MIS)

[Denicol, Niemi, Molnar, Rischke; PRD (2012)]

cf. talk by Riccardo Longo

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BDNK hydrodynamics: causal stable *[Kovtun; JHEP (2019)]* *[Bemfica, Disconzi, Noronha; PRD (2018)]* *[PRX (2022)]*

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based on:
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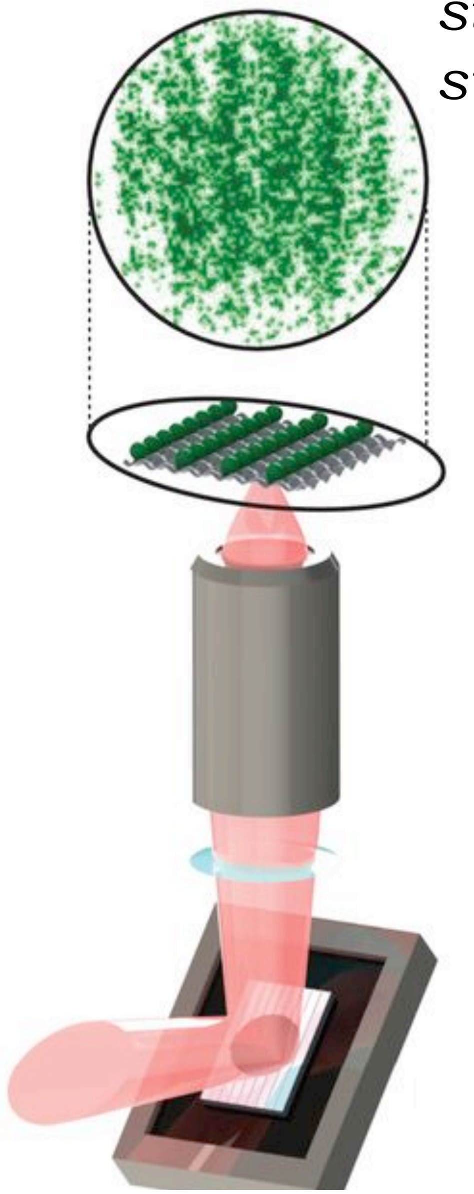
Hydrodynamics for ultracold atom experiment

Ultracold atom measurement: *Bad metallic transport in a cold atom Fermi-Hubbard system*

[Brown et al.; Science (2018)]



Lithium-6 atoms in optical lattice,
sinusoidal trap modulation
switched off instantly

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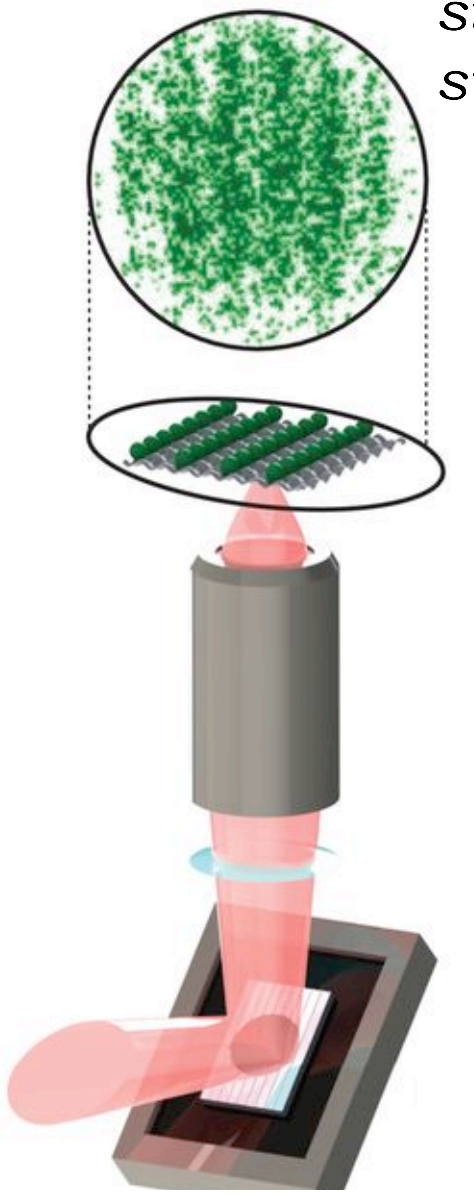


Lithium-6 atoms in optical lattice, sinusoidal trap modulation switched off instantly

Theory: Diffusion coefficient modified by quantum-statistical fluctuations (e.g. near critical points)

[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]
[Kovtun, Moore, Romatschke; PRD (2011)]

Hydrodynamics needs to be made *causal and stable* (e.g. MIS, DNMR, BDNK, …)

Include **quantum-statistical fluctuations** into a ***causal and stable*** version of hydrodynamics, **analyze experimental data with this version**:

How does diffusion change?

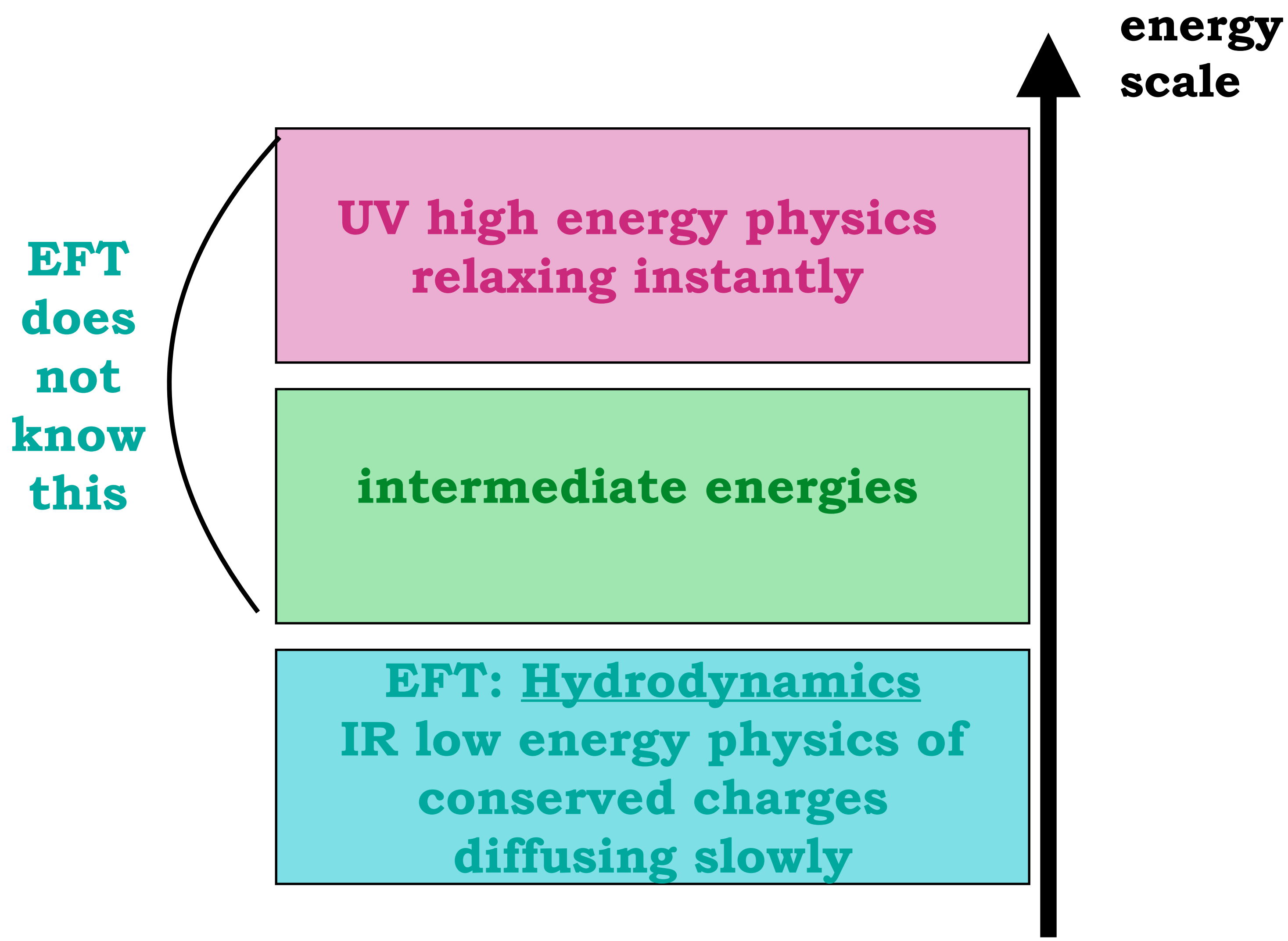
[Abbasi, Kaminski, Tavakol; PRL (2024)]

Outline

1. Motivation
2. **Causal stable hydrodynamics**
3. Fluctuating causal stable diffusion
4. Hazy visions
5. Discussion

Idea: causal stable hydrodynamics by renormalization

Effective Field Theory (EFT) needs to be regularized and renormalized



➡MIS, DNMR, BDNK are distinct renormalization schemes for hydrodynamics

What exactly is hydrodynamics?

Hydrodynamics

- **effective field theory (EFT)** of systems at late times and large distances
- conserved quantities survive

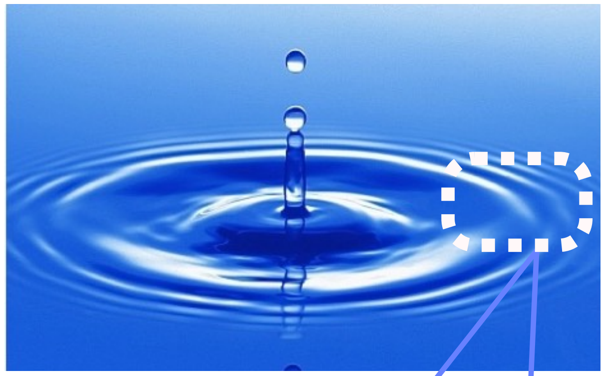
$$n(t,\vec{x}) \propto e^{-i\omega t + i\vec{k}\cdot\vec{x}_3}\, n(\omega,\vec{k})$$

- small **gradients**

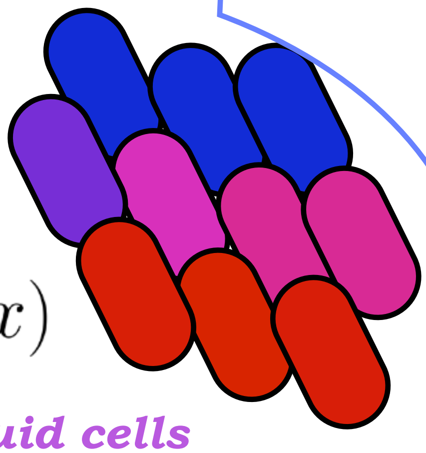
$$\partial_t\, e^{-i\omega t} = -i\omega\, e^{-i\omega t}$$

- large temperature

$$\frac{\omega}{T} \ll 1, \qquad \frac{|\vec{k}|}{T} \ll 1$$



$$T(t,\vec{x}) \equiv T(x)$$



Constitutive equations

$$\langle T^{\mu\nu}\rangle = \epsilon\, u^\mu u^\nu + P\,\Delta^{\mu\nu} + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + \dots$$

$$\langle j^\mu\rangle = n u^\mu + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + \dots$$

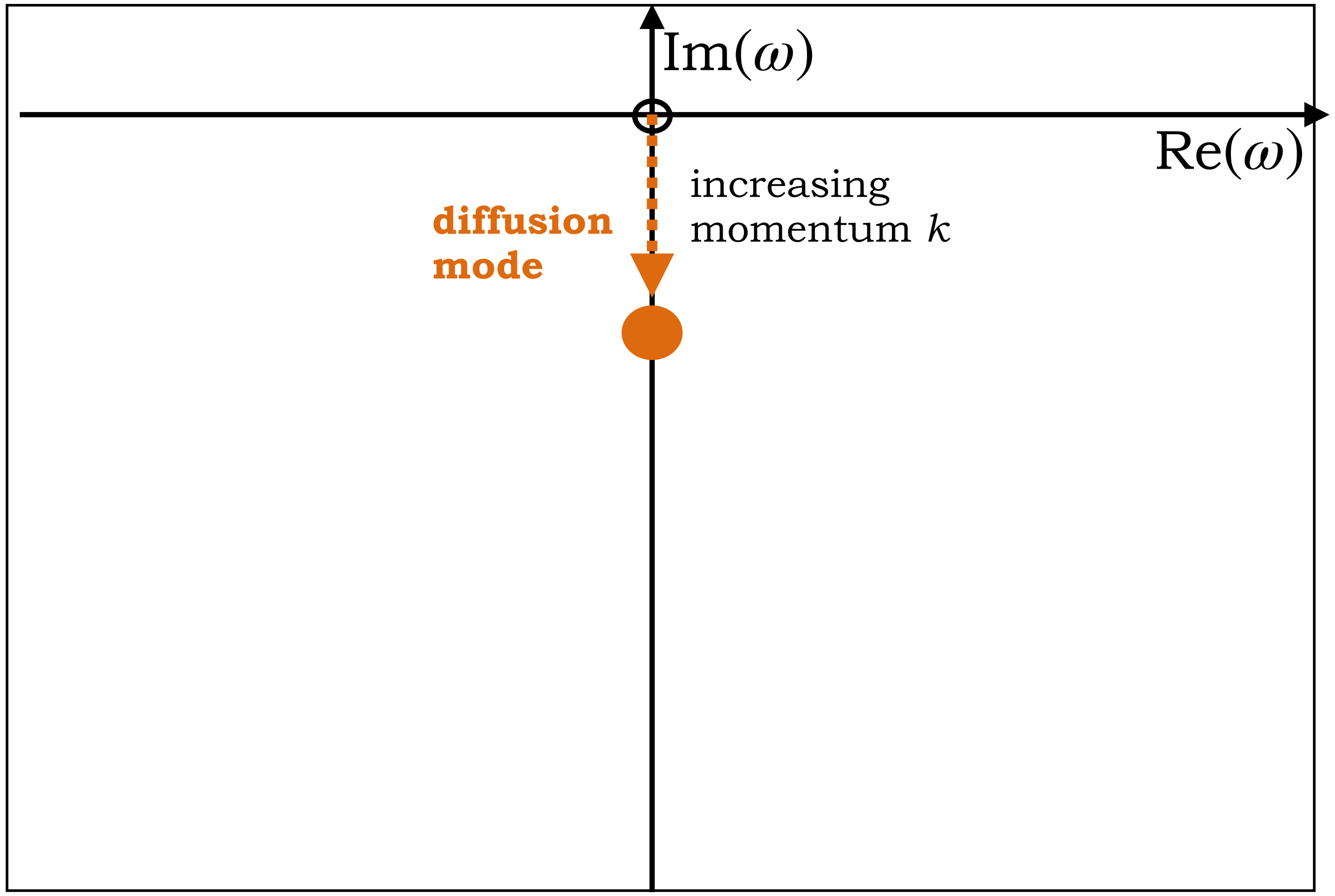
Conservation equations

$$\nabla_\mu \langle T^{\mu\nu}\rangle = 0$$

$$\nabla_\mu \langle j^\mu\rangle = 0$$

Hydrodynamic diffusion mode

Dispersion of eigenmodes in complex frequency plane



Consider one conserved charge n

$$\partial_t n + \nabla \cdot \mathbf{J} = 0\,, \qquad \mathbf{J} + D\,\nabla n = 0$$

Fick's law of diffusion:

$$\partial_t n - D\nabla^2 n = 0$$

Fourier transform $n(t,x) \propto e^{-i\omega t + ikx}\, n(\omega, k)$
to read off eigen-frequency:

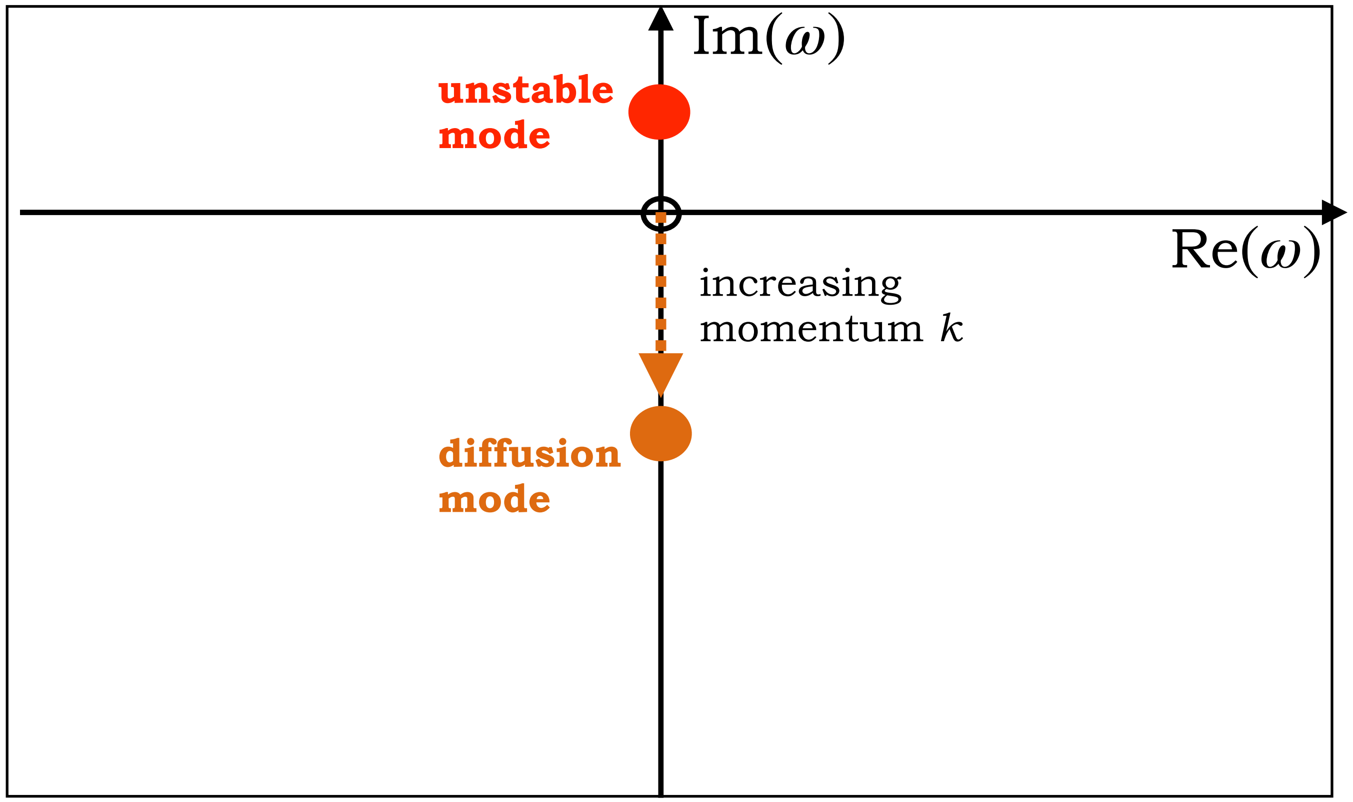
$$\omega = -\, iDk^2$$

diffusion mode

Differential equation turned into algebraic equation by relations like $\partial_t\, e^{-i\omega t} = -i\omega\, e^{-i\omega t}$

Unstable diffusion after Lorentz boost

Dispersion of eigenmodes in complex frequency plane *[Jain, Kovtun; JHEP (2023)]*



Fick's law of diffusion:

$$\partial_t n - D\nabla^2 n = 0$$

$$\omega = -\, iDk^2$$

Lorentz boost by velocity v_0:

$$\omega \to \frac{1}{\sqrt{1-v_0^2}}\left(\omega - kv_0\cos\theta\right), \qquad k_i \to \frac{1}{\sqrt{1-v_0^2}}\left(k\cos\theta - \omega v_0\right)\frac{v_i^0}{v_0} + k_j\left(\delta_i^j - \frac{v_i^0 v_0^j}{v_0^2}\right)$$

$$k^2 \to \frac{1}{1-v_0^2}\left(k\cos\theta - \omega v_0\right)^2 + k^2\sin^2\theta,$$

symmetry transformation, should not change physics

$$\omega = kv_0\cos\theta - iDk^2\sqrt{1-v_0^2}\left(1 - v_0^2\cos^2\theta\right) + \dots$$

boosted diffusion mode

$$\omega = i\frac{\sqrt{1-v_0^2}}{v_0^2 D} + \dots .$$

additional unstable gapped mode

➡ **This hydrodynamic formalism for diffusion is unstable.**

Causal stable hydrodynamics

Stability and causality

- No exponential growth
- response follows cause

$$\mathrm{Im}\,\omega(k) \leq 0$$

- no superluminal response

$$\left| \mathrm{Re}\,\frac{\omega(k)}{k} \right| \leq 1$$

textbook hydrodynamics violates both inequalities!

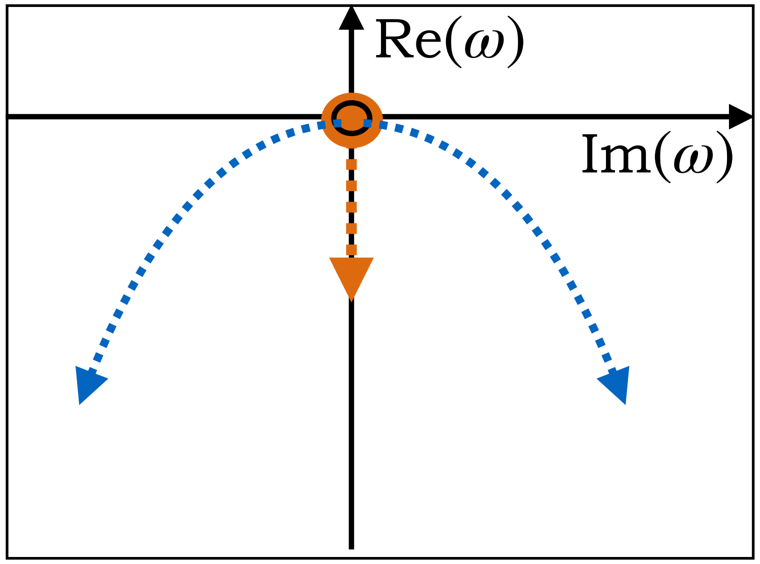
Examples

Diffusion mode

$$\omega(k) = -iDk^2 + \mathcal{O}(3)$$

Sound modes

$$\omega(k) = \pm v_s k - i\Gamma k^2 + \mathcal{O}(3)$$



Complex frequency plane

Versions of causal stable hydrodynamics

- Mueller-Israel-Stewart (MIS) and DNMR
- Bemfica-Disconzi-Noronha-Kovtun (BDNK)
- Schwinger-Keldysh (SK)

Causal stable hydrodynamics

Versions of causal stable hydrodynamics

- Mueller-Israel-Stewart (MIS) and DNMR:
  - **fluxes of conserved charges evolve independently** of the respective densities and relax after a **relaxation time** τ
- Bemfica-Disconzi-Noronha-Kovtun (BDNK)
  - **redefinitions of hydrodynamic fields** to introduce the **relaxation time** τ into the hydrodynamic equations
- Schwinger-Keldysh (SK) EFT of hydrodynamics
  - **covariant renormalized** generating functional

➡ all versions provide UV-regularization of the EFT (hydrodynamics)

Hydrodynamic modes

Interacting many-body systems at large temperature T have collective excitations, damped eigenmodes, with specific dispersion relations :

(assuming rotation invariance: $k \equiv |\vec{k}|$)

Sound modes (gapless)

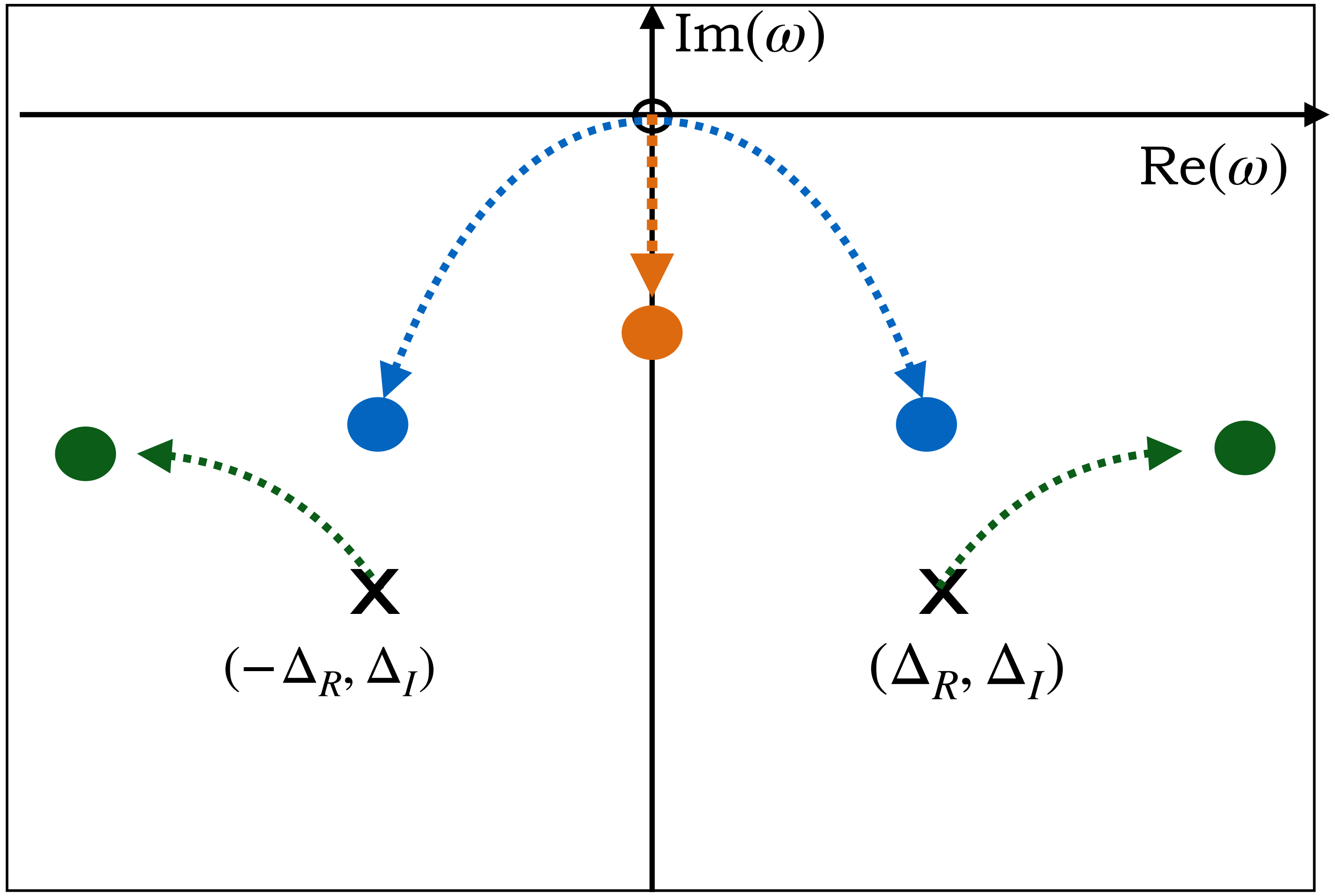
$$\omega(k) = \pm v_s k - i\Gamma k^2 + \mathcal{O}(3)$$

Diffusion mode (gapless)

$$\omega(k) = -iDk^2 + \mathcal{O}(3)$$

Non-hydrodynamic UV modes

$$\omega(k) = \underbrace{\pm\Delta_R - i\Delta_I}_{\text{gap}} + v_0 k - i\Gamma_0 k^2 + \mathcal{O}(3)$$



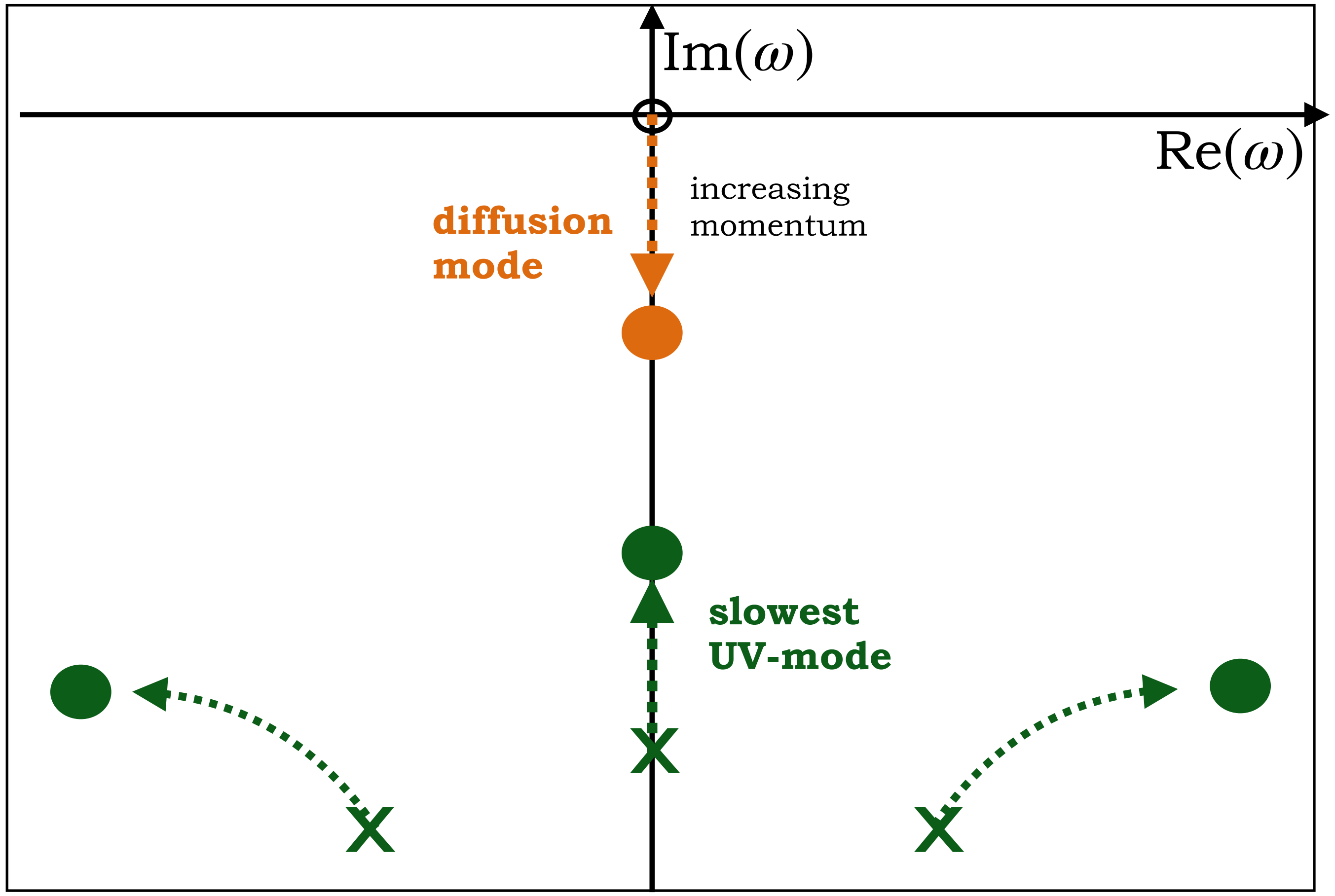
Complex frequency plane

➡ there are hydrodynamic modes (gapless) and non-hydrodynamic UV modes (gapped)

Causal stable diffusion

[Abbasi, Kaminski, Tavakol; PRL (2024)]

Dispersion of eigenmodes in complex frequency plane



Consider one conserved charge n and ***relaxation time*** τ:

$$\partial_t n + \nabla \cdot \mathbf{J} = 0\,, \quad \boxed{\tau\,\partial_t \mathbf{J}} + \mathbf{J} + D\,\nabla n = 0$$

Fick's law of diffusion ***(UV-regulated)***:

$$\boxed{\tau \partial_t^2 n} + \partial_t n - D \nabla^2 n = 0$$

Fourier transform to read off eigen-frequencies:

$$\boxed{\omega_{1,2} = -\frac{i}{2\tau}\left(1 \mp \sqrt{1 - 4\tau D \mathbf{k}^2}\right)}$$

diffusion mode and
slowest UV-mode

This is the standard formulation of relativistic hydrodynamics used in numerical codes for heavy ion collisions (Mueller-Israel-Stewart), one may also think of this as Hydro+

concise summary in section V. B of white paper [Sorensen et al.; arXiv:2301.13253]

Outline

Objective

How do nonlinear ***quantum-statistical fluctuations*** modify diffusion when taking into account the ***slowest UV-mode***?

[Abbasi, Kaminski, Tavakol; PRL (2024)]

Fluctuations (nonlinear in amplitudes)

Hydrodynamics involves two expansions into

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Fick's law of diffusion ***(UV-regulated)***:

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Self-interactions enter through dependence of diffusion coefficient on charge fluctuations:

$$D(n) = D + \lambda_D n + \frac{\lambda_D'}{2} n^2 + \ldots$$

Two formalisms to include fluctuations

linear diffusion

linearized in hydrodynamic fields

(e.g., energy density ϵ ~ temperature T)

nonlinear diffusion via effective action from exponentiated e.o.m. Martin-Siggia-Rose formalism (MSR)

write stochastic differential equations as a field theory formulated using path integrals

[Martin, Siggia, Rose; PRA (1973)]

nonlinear diffusion via effective action via Schwinger-Keldysh formalism (SK)

effective field theory for dissipative hydrodynamics

[Crossley, Glorioso, Liu; JHEP (2017)]

MSR is used here.

[Abbasi, Kaminski, Tavakol; PRL (2024)]

adding one regulating UV-mode

charge diffusion

SK was used in

[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

no regulating UV-mode

heat diffusion

Method: perturbative calculation from MSR

[Abbasi, Kaminski, Tavakol; PRL (2024)]

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Leading to the nonlinear equation of motion:

$$\tau \partial_t^2 n + \partial_t n - \nabla^2 \left(D\, n + \frac{\lambda_D}{2} n^2 + \frac{\lambda_D'}{6} n^3 \right) = 0$$

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Exponentiate stochastic version of this equation to obtain path integral, from which the effective action can be read:

[Martin, Siggia, Rose; PRA (1973)]

$$\mathcal{L} = i\,T\sigma\,\nabla n_a\, C\,\nabla n_a - n_a\left(\tau\partial_t^2 n + \partial_t n - D\nabla^2 n\right)$$

$$+\, iT\chi\lambda_\sigma\, n\nabla n_a C\nabla n_a + \frac{\lambda_D}{2}\,\nabla^2 n_a\, n^2 + \frac{1}{2} iT\chi\lambda_\sigma'\, n^2\nabla n_a C\nabla n_a + \frac{\lambda_D'}{6}\,\nabla^2 n_a\, n^3$$

with conductivity $\sigma(n) = \sigma + \chi\lambda_\sigma\delta n + \frac{1}{2}\chi\lambda_\sigma'\delta n^2$ and $C = \left(\frac{i\partial_t}{2T}\right)\coth\left(\frac{i\partial_t}{2T}\right)$

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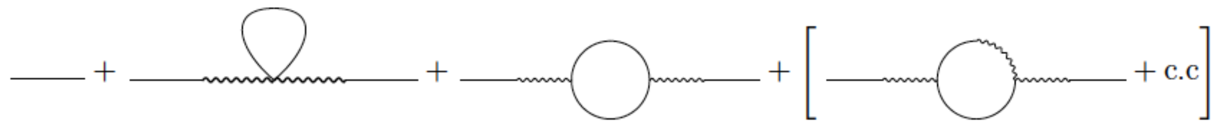
Perform perturbation theory computation to one-loop order, like done in particles physics (e.g. QED).

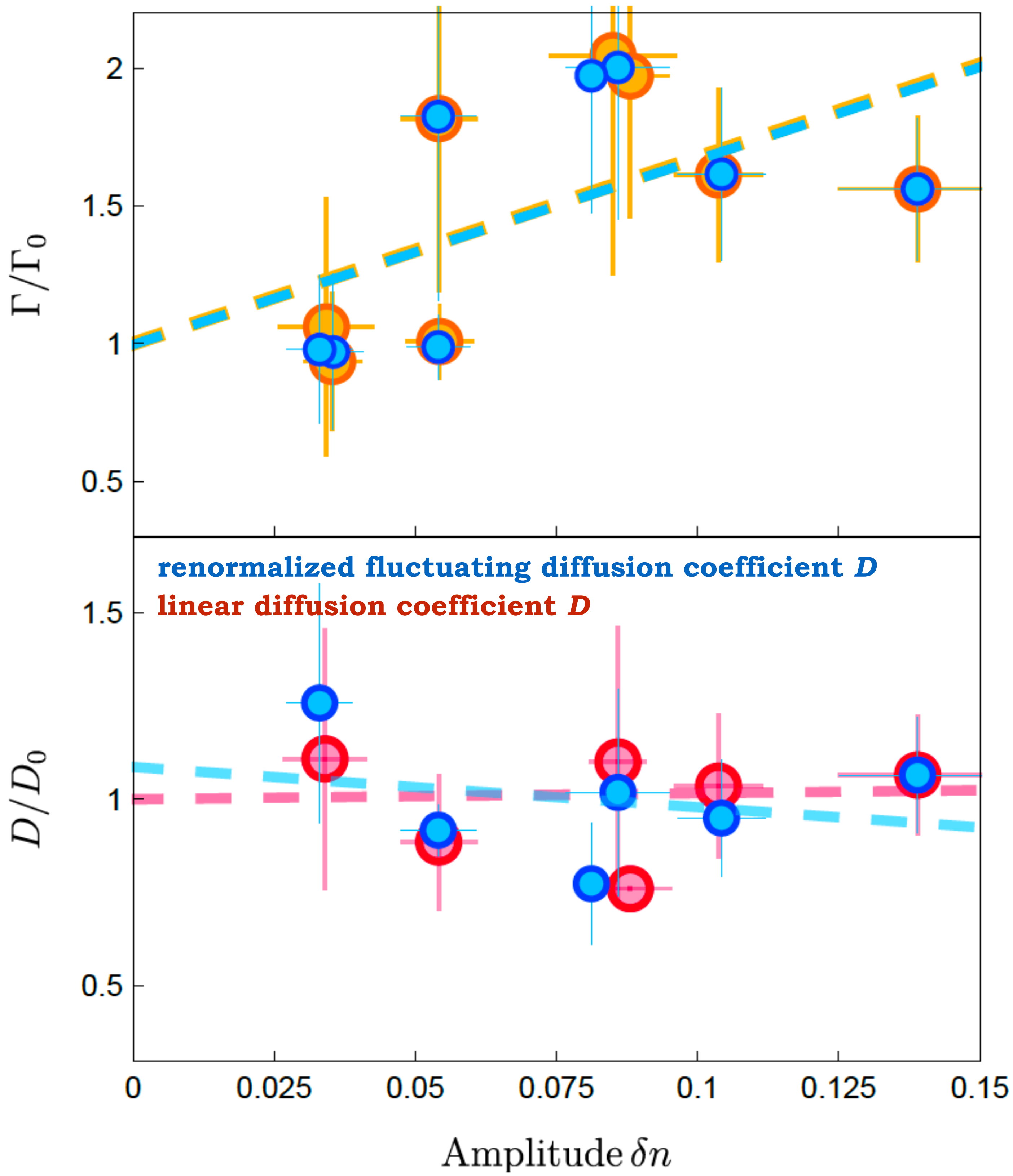
$G^{(0)}_{nn_a}(p)(-C(p))G^{(0)}_{n_a n}(p) =$



Analyzing data with causal stable fluctuating theory of diffusion

[Abbasi, Kaminski, Tavakol; PRL (2024)]

Ultracold atom data from
[Brown et al.; Science (2018)]



➡**diffusion coefficient *D* gets renormalized by fluctuations (quantum and statistical)**

➡**assuming that large fluctuations exist: more accurate *D* and strength of non-linear effects**

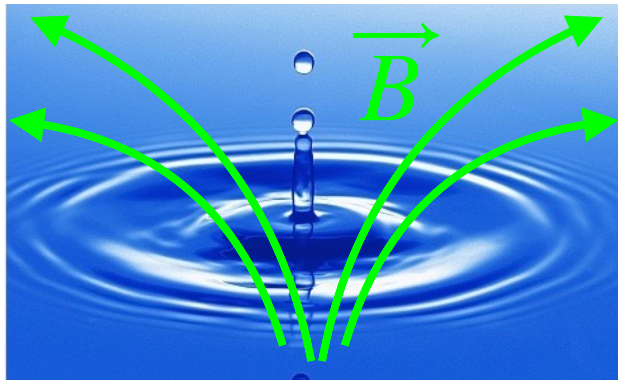
$$\lambda_D = \frac{dD(n)}{dn}$$

➡**our non-linear theory should be tested in future experiments**

Outline

1. Motivation
2. Causal stable hydrodynamics
3. Fluctuating causal stable diffusion
4. **Hazy visions**
5. Discussion

Effective Field Theories for the EIC?



Dynamical vs. External

- hydrodynamics in **external** electric/ magnetic fields
- generalized to **dynamical** electric/ magnetic fields obeying Maxwell equations (=Magneto-HydroDynamics)

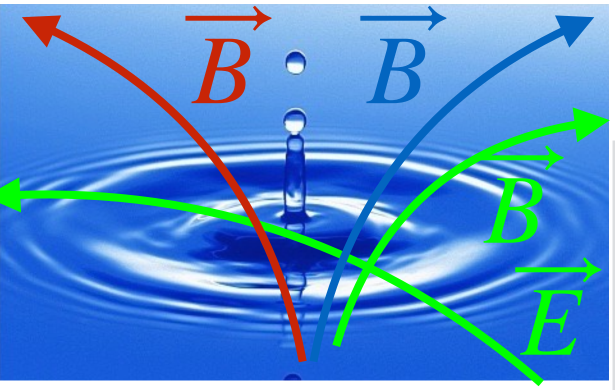
[Hernandez, Kovtun; JHEP (2017)]

cf. [Grozdanov, Hofman, Iqbal; PRD (2016)]

Idea

- EIC probes/creates densely packed gluons, pushing towards the color glass condensate (CGC)
  ➡effective field theory needed?**ChromoHydro?**
- Transition/crossover into CGC?
  ➡fluctuations needed?

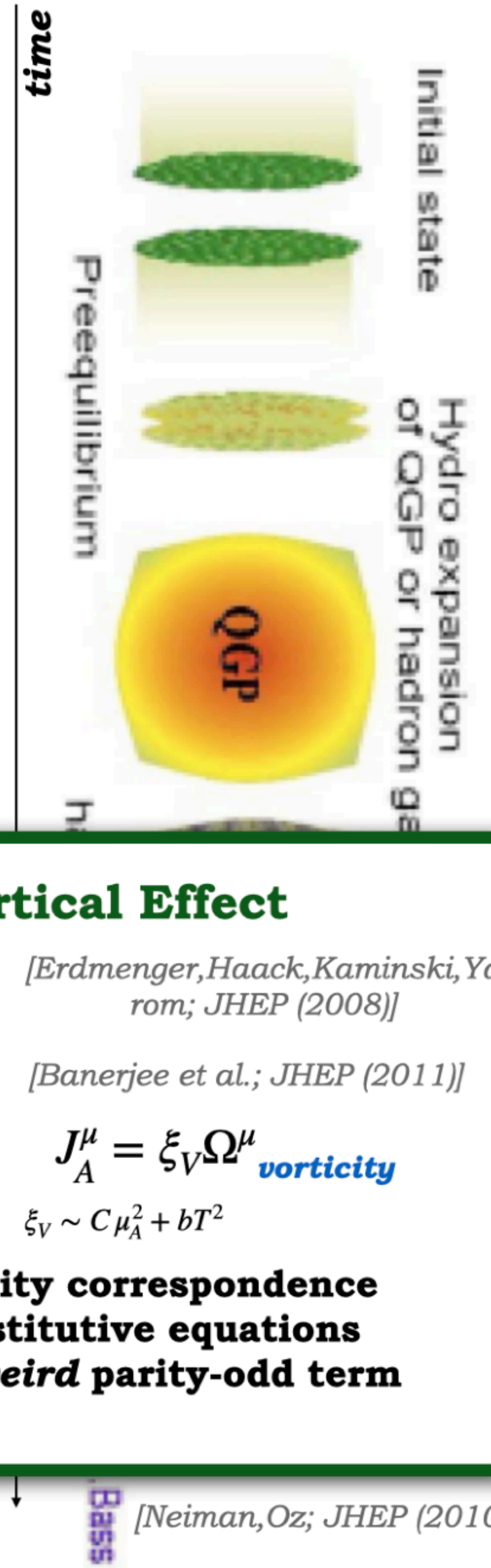
Chromo-Hydrodynamics



A brief history of Chromo-Hydrodynamics

- A new approach to non-Abelian hydrodynamics
  [Fernandez-Melgarejo, Rey, Surowka; JHEP (2017)]
- applicable to short-time scale color phenomena in the QGP; gluon polarization tensor derived; plasma instabilities; no ChromoMHD
  [Manuel, Mrowczynski; PRD (2006)]
- Yang-Mills magneto-fluid unification
  [Bambah, Mahajan, Mukku; PRL (2006)]
- Chromohydrodynamics of the QGP
  [Mrowczynski; PLB (1988)]
- Relativistic chromohydrodynamics and Yang-Mills Vlasov plasma
  [Holm, Kupershmidt; Phys.Lett.A (1984)]
- Kinetic Theory for Plasmas with Non-Abelian Interactions *[Heinz; PRL (1983)]*

A winning team: hydrodynamics and holography in parallel



HYDRODYNAMICS & THERMODYNAMICS

Chiral Magnetic Effect (CME) from chiral anomaly

[Kharzeev; PRC (2004)]
[Son,Surowka; PRL (2009)]
[Neiman,Oz; JHEP (2010)]

$$J^\mu_A = \xi_B B$$

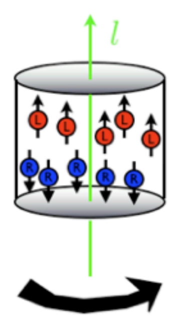
hydro and holo in parallel

HOLOGRAPHY

CME far from equilibrium, strong B

- **non-expanding plasma** *[Gosh,Grieninger,Landsteiner,Morales-Tejera; PRD (2021)]*
- **expanding plasma** *[Cartwright,Kaminski,Schenke; PRC (2022)]*

Chiral Vortical Effect



[Erdmenger,Haack,Kaminski,Yarom; JHEP (2008)]

[Banerjee et al.; JHEP (2011)]

$$J^\mu_A = \xi_V \Omega^\mu \quad \textit{vorticity}$$

$$\xi_V \sim C\mu_A^2 + bT^2$$

- **fluid/gravity correspondence**
- **gives constitutive equations**
- **contain *weird* parity-odd term**

[Neiman,Oz; JHEP (2010)]

DISCLAIMER: More balanced review in my Section 5.2 on Hydrodynamics in White Paper [Sorensen et al.; Prog.Part.Nucl.Phys. (2024)]

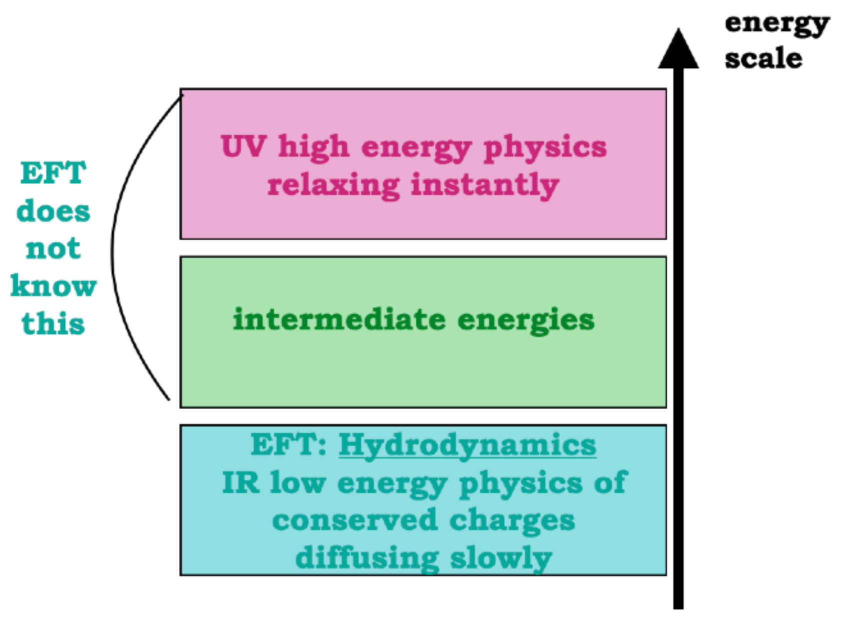
Discussion

Summary

- hydrodynamics was developed as an **effective field theory** (derived from generating functional, renormalized, causal, stable, fluctuating, …)
- **holography** (AdS/CFT) was developed in parallel
  - **cross-checks** (in common low-energy regime)
  - **discovery tool** (chiral vortical effect, far from equilibrium, …)

Outlook

- **effective field theories** for the EIC (chromohydrodynamics?)
- **versions of hydrodynamics** for the EIC (hybrids?)
  cf. talk by Austin Baty (system size study, pA vs. eA)
- **holographic models** for EIC physics?
  cf. talk by Alex Buchel ?



Our upcoming ECT* Workshop:
March 24-28, 2025 (with Gursoy/Kharzeev/Landsteiner)
Holographic perspectives on chiral transport and spin dynamics

APPENDIX

APPENDIX: Discussion points

- key properties of EIC: polarization, gluon dense, ...
- depolarization due to quantum fluctuations
- collective effects in small systems?
- EFT for polarized particles *interacting with medium?*
- EIC and hydrodynamics?
- EFTs for EIC?
- fluctuations important near phase transitions
- renormalized transport coefficients
- ...

APPENDIX: Hydrodynamic modes

Interacting many-body systems at large temperature T have collective excitations, damped **eigenmodes**, with specific dispersion relations :

(assuming rotation invariance: $k \equiv |\vec{k}|$)

Sound modes (gapless)

$$\omega(k) = \pm v_s k - i\Gamma k^2 + \mathcal{O}(3)$$

Diffusion mode (gapless)

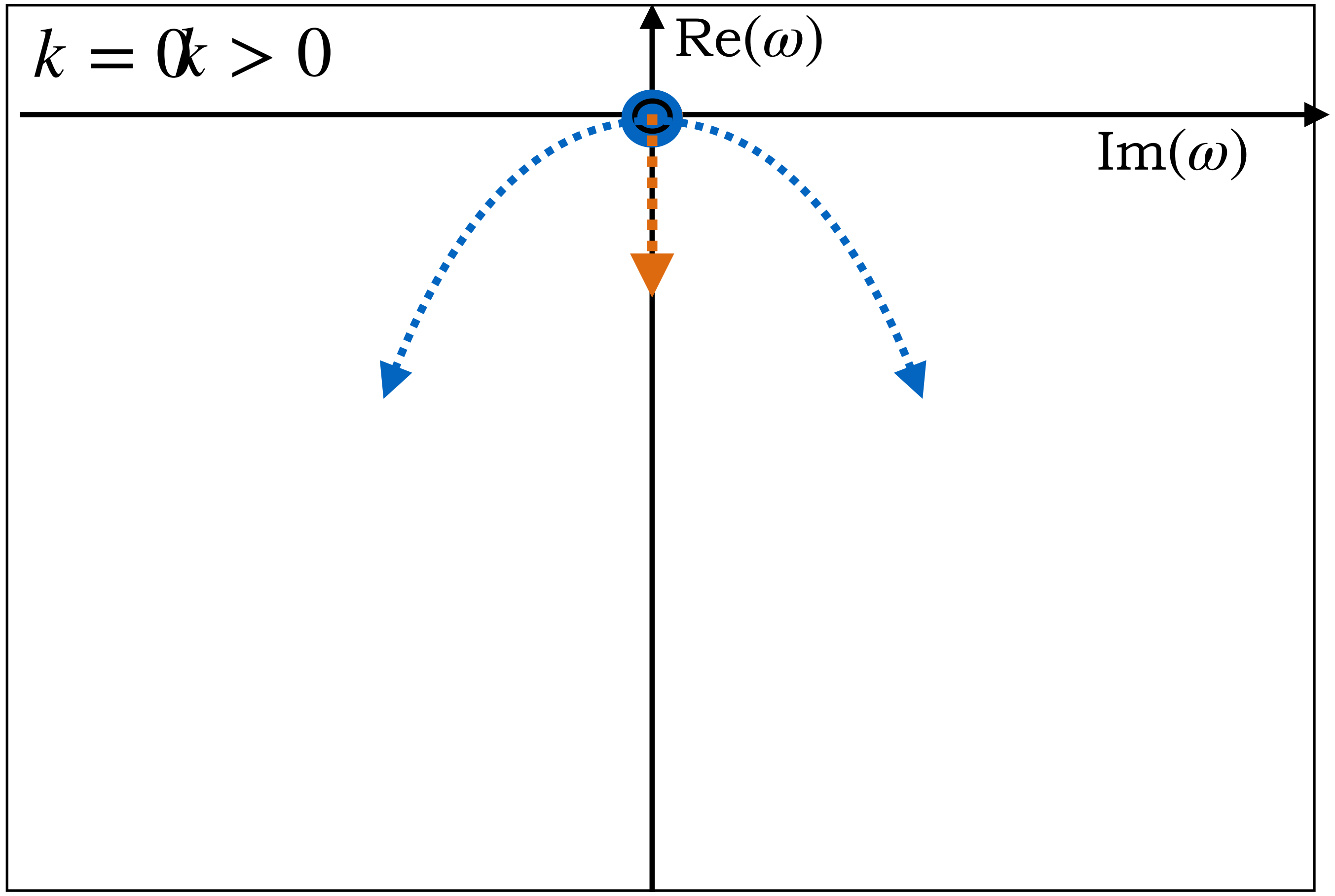
$$\omega(k) = -iDk^2 + \mathcal{O}(3)$$

$$\partial_t n - D\nabla^2 n = 0$$

linear equation of motion for conserved quantity

$$\mathscr{P}\, G^R = \delta$$

$$G^R \propto \mathscr{P}^{-1} \propto \frac{1}{\partial_t - D\partial_x^2 + \mathcal{O}(3)} \propto \frac{1}{\omega + iDk^2 + \mathcal{O}(3)}$$

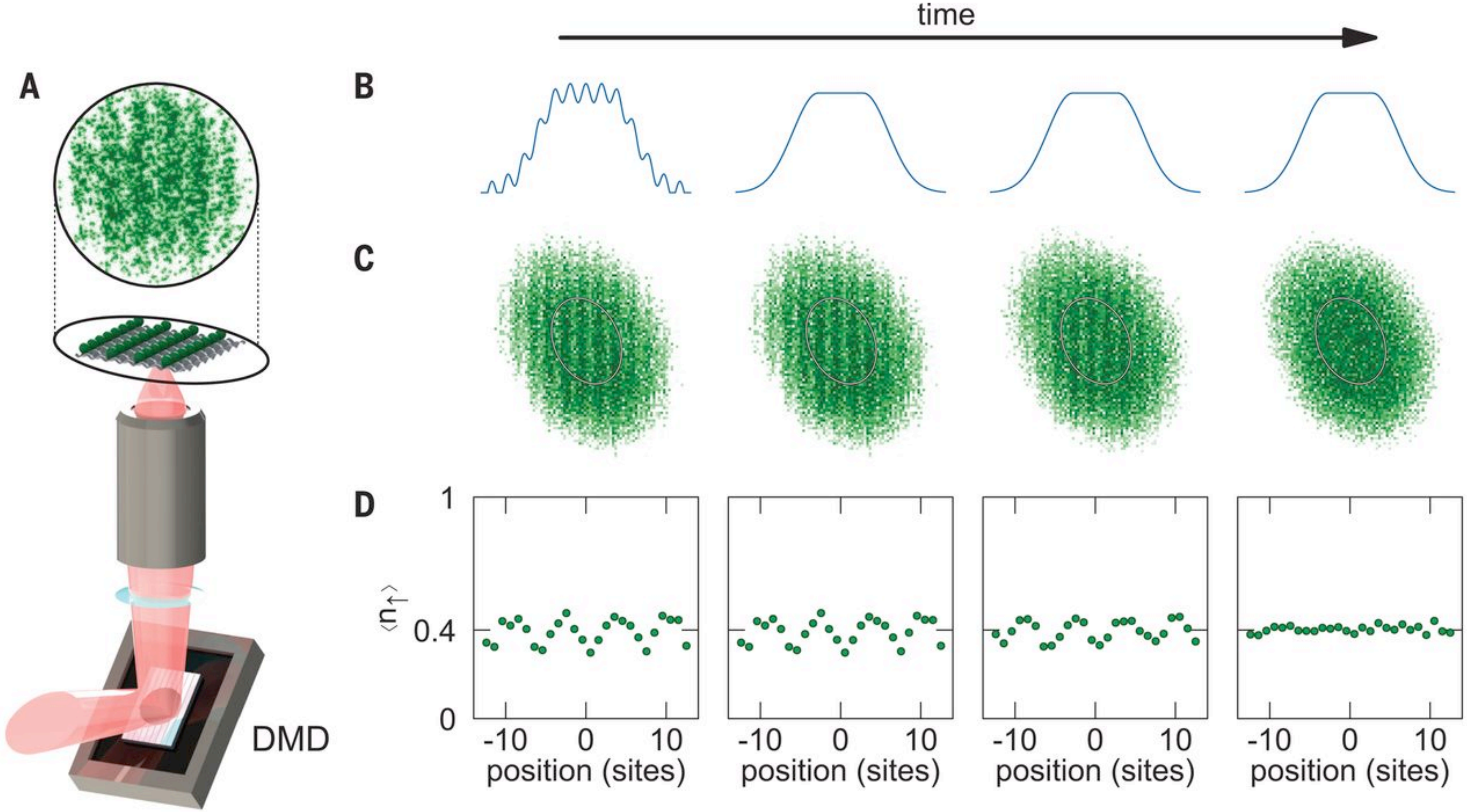


Complex frequency plane

APPENDIX: Experiment

Ultracold atom measurements: *Bad metallic transport in a cold atom Fermi-Hubbard system*
Lithium-6 atoms in optical lattice, sinusoidal trap modulation switched off instantly

[Brown et al.; Science (2018)]



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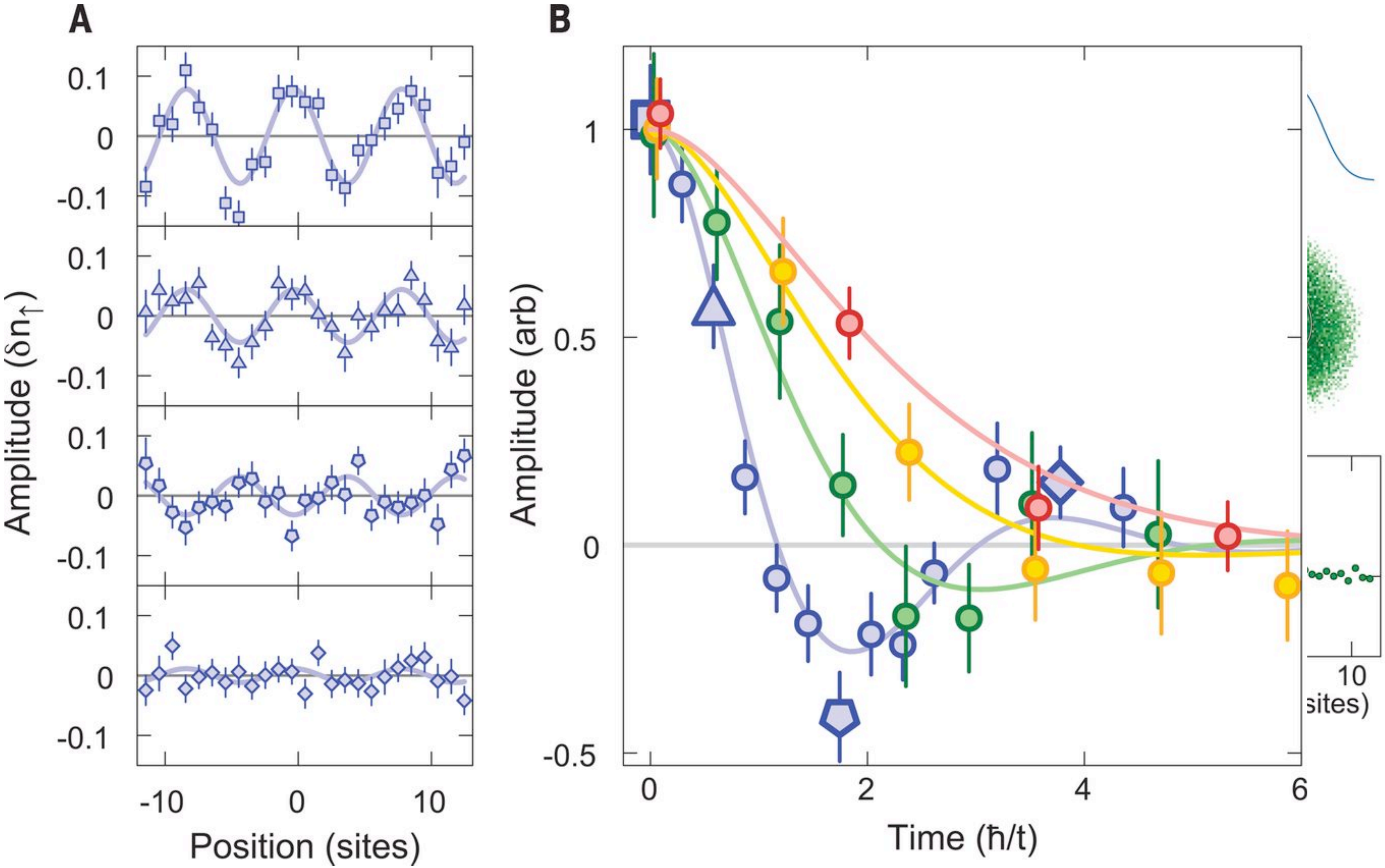

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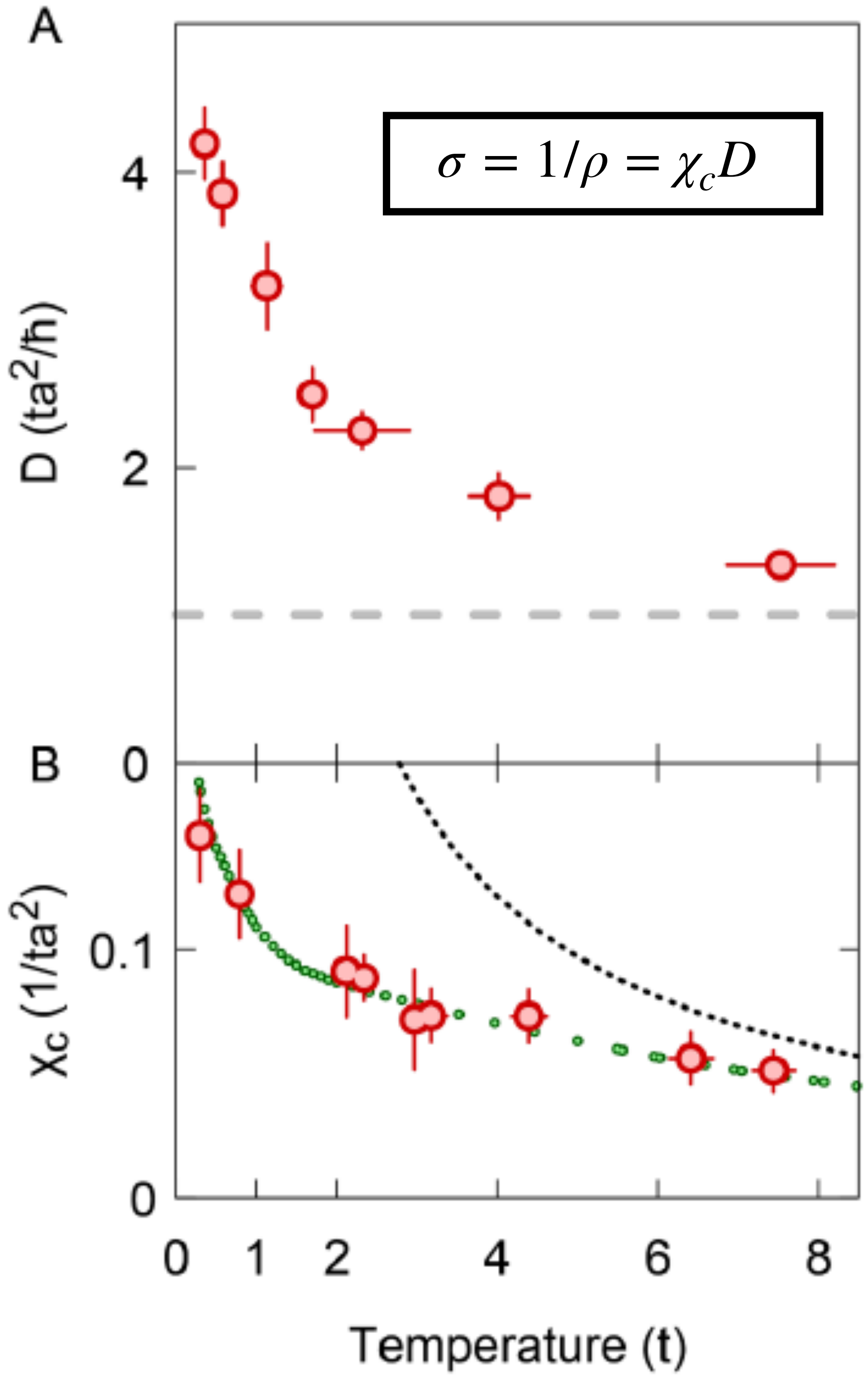


Diffusion coefficient modified by quantum-statistical fluctuations (e.g. near critical points)

[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

Hydrodynamics as an effective field theory

[Jensen, Kaminski, Kovtun,Meyer,Ritz,Yarom.; PRL (2012)]
[Banerjee et al. JHEP (2012)]
[Crossley, Glorioso, Liu; JHEP (2017)]
[Haehl, Loganayagam, Rangamani; JHEP (2015)]

BUT: hydrodynamics needs to be regulated by UV-mode(s) to be *causal and stable*
(e.g. Mueller-Israel-Stewart theory, BDNK, ...)

[Hiscock & Lindblom; PRD (1985)]
[Bemfica, Disconzi, Noronha; PRD (2018)] *[PRX (2022)]*
[Hoult,Kovtun; JHEP (2020)] *[Kovtun; JHEP (2019)]*
...
[Abbasi, Kaminski, Tavakol; PRL (2024)]

APPENDIX: Method of Martin-Siggia-Rose

nonlinear diffusion via effective action from exponentiated e.o.m. Martin-Siggia-Rose formalism (MSR)

write stochastic differential equations as a field theory formulated using path integrals

[Martin, Siggia, Rose; PRA (1973)]

Idea:

$$\langle \mathcal{O} \rangle \sim e^{e.o.m.}$$

Stochastic differential equation (e.o.m.):

$$\partial_t x(t) = F(x(t), t) + \xi(x(t), t),$$

Noise correlation:

$$\langle \xi(x,t)\xi(x',t')\rangle = G(x,t,x',t').$$

Observables averaged over solutions of this stochastic differential equation may be written:

$$\langle \mathcal{O}[x(t)]\rangle = \int \mathcal{D}[x,\tilde{x}]\, \mathcal{O}[x(t)]\, e^{-S[x,\tilde{x}]},$$

$$S[x,\tilde{x}] = \int_t i\tilde{x}(t)\,[\partial_t x(t) - F(x(t),t)] + \frac{1}{2}\int_{t,t'} G(x(t),t,x(t'),t')\tilde{x}(t)\tilde{x}(t').$$

APPENDIX: Method

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

Self-interactions enter through dependence of diffusion coefficient on charge fluctuations:

$$D(n) = D + \lambda_D n + \frac{\lambda'_D}{2} n^2$$

Leading to the nonlinear equation of motion:

$$\tau \partial_t^2 n + \partial_t n - \nabla^2 \left(D\, n + \frac{\lambda_D}{2} n^2 + \frac{\lambda'_D}{6} n^3 \right) = 0$$

Note: corrections to $\tau(n) = \tau + \lambda_{\tau,1} n + \lambda_{\tau,2} n^2 + \ldots$ **contribute to higher order only**

Exponentiate stochastic version of this equation to obtain path integral, from which the effective action can be read: *[Martin, Siggia, Rose; PRA (1973)]*

$$\mathcal{L} = i T \sigma\, \nabla n_a\, C\, \nabla n_a - n_a \left(\tau \partial_t^2 n + \partial_t n - D \nabla^2 n \right)$$
$$+\, i T \chi \lambda_\sigma\, n \nabla n_a C \nabla n_a + \frac{\lambda_D}{2} \nabla^2 n_a\, n^2 + \frac{1}{2} i T \chi \lambda'_\sigma\, n^2 \nabla n_a C \nabla n_a + \frac{\lambda'_D}{6} \nabla^2 n_a\, n^3$$

with conductivity $\sigma(n) = \sigma + \chi \lambda_\sigma \delta n + \frac{1}{2} \chi \lambda'_\sigma \delta n^2$ and $C = \left(\frac{i\partial_t}{2T}\right) \coth\left(\frac{i\partial_t}{2T}\right)$

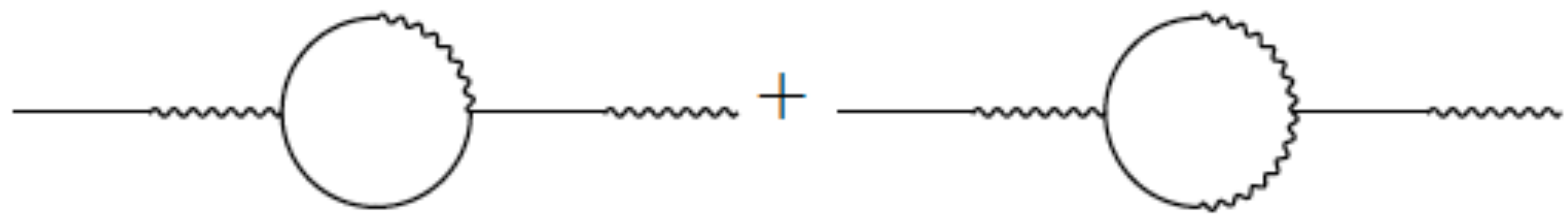
Perform perturbation theory computation to one-loop order, like done in particle physics (e.g. QED).

[Abbasi, Kaminski, Tavakol; arXiv:2212.11499]

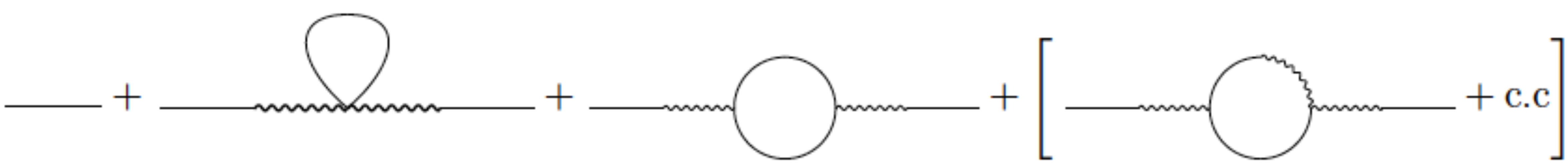
Perform perturbation theory computation to one-loop order, like done in particle physics (e.g. QED).

$$G_{nn_a}(p) = G^{(0)}_{nn_a}(p) + G^{(0)}_{nn_a}(p)(-\Sigma(p))G^{(0)}_{nn_a}(p) = \frac{1}{\omega + iD_0\mathbf{k}^2 - i\tau\,\omega^2 + \Sigma(\omega,\mathbf{k})}$$

$$G^{(0)}_{nn_a}\Sigma(p)G^{(0)}_{nn_a} =$$



$$G^{(0)}_{nn_a}(p)(-C(p))G^{(0)}_{n_an}(p) =$$



[Abbasi, Kaminski, Tavakol; PRL (2024)]

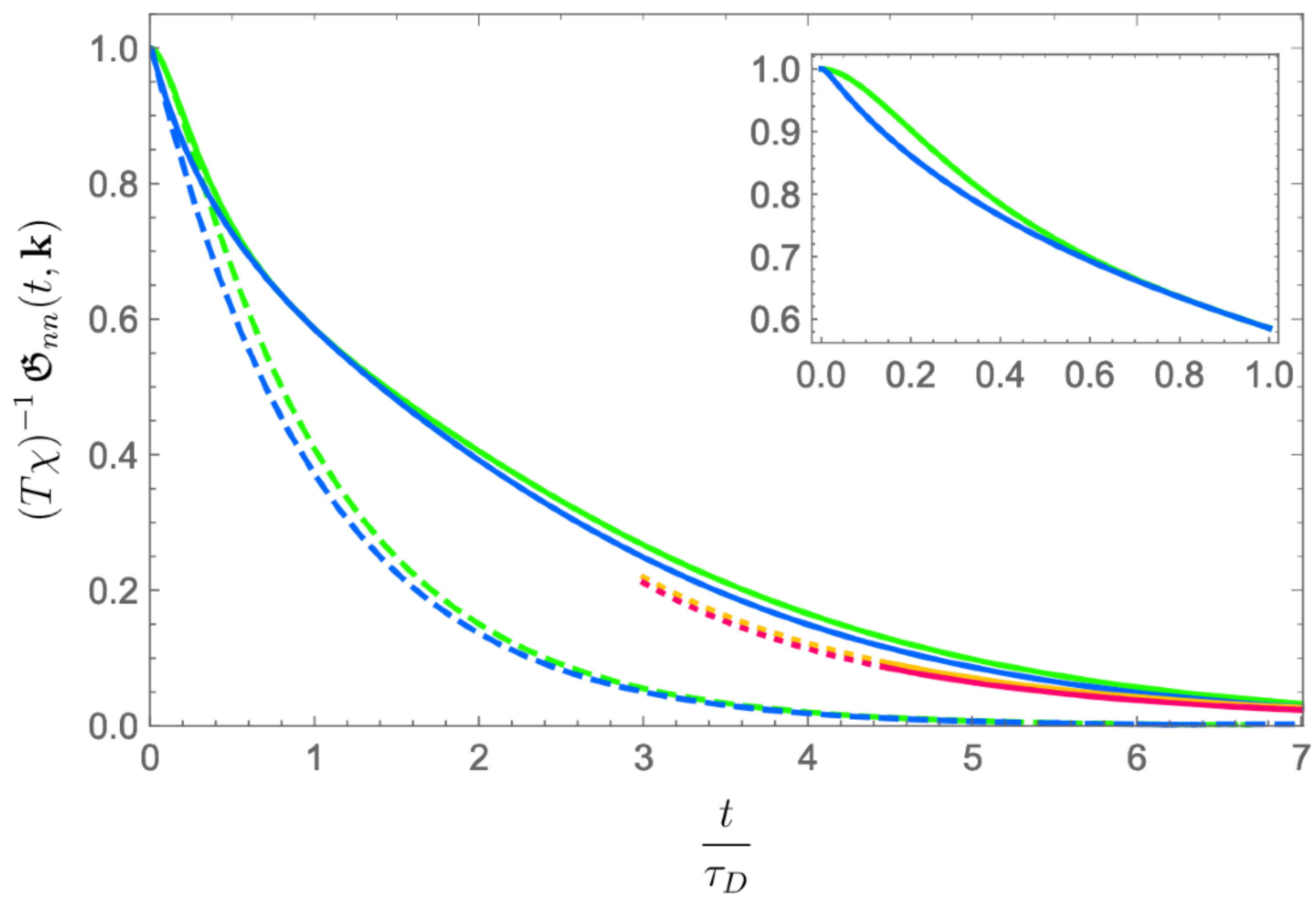


Figure 2. Solid blue and green curves show $\mathfrak{G}_{nn}^{(0)+(1)}$ while dashed curves correspond to $\mathfrak{G}_{nn}^{(0)}$. With blue and green, we illustrate distinct values of $r = \tau_{UV}/\tau_D$: 1/100 and 1/10, respectively. The red and yellow curves show the long-time-tail of $\mathfrak{G}_{nn}^{(0)+(1)}$ for these two values of r.

Applied to Bjorken expanding plasma

$$\Delta\langle n(\tau_p)\rangle = a\,T\chi^2\mu\,\frac{\lambda_D^2}{D^2}\frac{1}{(D\tau_p)^{3/2}}\left(1-\frac{11}{8}\frac{\tau}{\tau_p}+\cdots\right),$$

[Abbasi, Kaminski, Tavakol; PRL (2024)]

effect of the UV-regulator on the late-time nonlinear correction to the single charge density, $\Delta\langle n(\tau_p)\rangle$,
as compared to non-fluctuating Bjorken flow

[Martinez, Schaefer; PRC (2018)]

Method: Effective formalism for hydrodynamic fluctuations

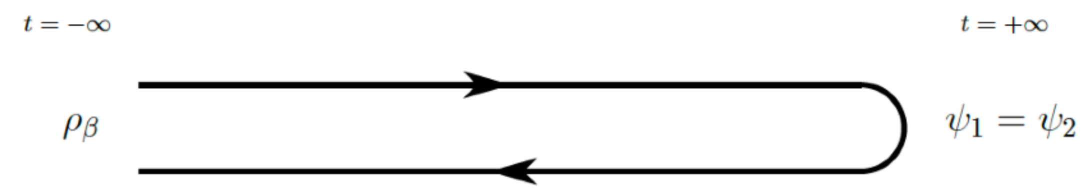
Supplemental Material of [Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

Goal is to compute correlator:

$$\langle \varepsilon(t,x)\varepsilon(t',x')\cdots\rangle_\beta \equiv \mathrm{Tr}\Big(\rho_\beta\,\varepsilon(t,x)\varepsilon(t',x')\cdots\Big) \qquad \rho_\beta = e^{-\beta H} / \mathrm{Tr}\, e^{-\beta H}$$

Generating functional:

$$Z[A^1_\mu, A^2_\mu,] \equiv \mathrm{Tr}\left(U[A_1]\rho_\beta U[A_2]^\dagger\right) \qquad Z[A^1_\mu, A^2_\mu] = \int D\psi_1 D\psi_2\, e^{iS[\psi_1,A_1]-iS[\psi_2,A_2]}$$



Constraints on effective action:

$$\begin{aligned}
& Z[A_\mu, A_\mu] = 1\,, \\
\text{reflection:} \quad & Z[A^1_\mu, A^2_\mu] = Z^*[A^2_\mu, A^1_\mu]\,, \\
\text{gauge invariance:} \quad & Z[A^1_\mu, A^2_\mu] = Z[A^1_\mu + \partial_\mu\lambda^1, A^2_\mu + \partial_\mu\lambda^2]\,, \\
\text{KMS condition:} \quad & Z[A^1_\mu, A^2_\mu] = Z[A^1_\mu(-t, x_{PT}), A^2_\mu(-t-i\beta, x_{PT})]\,,
\end{aligned}$$

Local effective action I: $Z[A^1_\mu, A^2_\mu] = \int D\varphi_1 D\varphi_2\, e^{iI[B^1_\mu, B^2_\mu]}$ $\qquad B_\mu = A_\mu + \partial_\mu\varphi$

Auxiliary fields: $\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2)\,, \qquad \varphi_a = \varphi_1 - \varphi_2$

Most general isotropic quadratic action:

$$\beta\mathcal{L}_2 = c\dot\varphi_r\dot\varphi_a + \kappa\dot\varphi_r\nabla^2\varphi_a + iT\tilde\kappa(\nabla\varphi_a)^2 + \cdots$$

$$\begin{aligned}
\mathcal{L} = {} & iT^2\kappa(\nabla\varphi_a)^2 - \varphi_a(\dot\varepsilon - D\nabla^2\varepsilon) \\
& + \nabla^2\varphi_a\left(\frac{\lambda}{2}\varepsilon^2 + \frac{\lambda'}{3}\varepsilon^3\right) + icT^2(\nabla\varphi_a)^2(\tilde\lambda\varepsilon + \tilde\lambda'\varepsilon^2) \\
& + \cdots,
\end{aligned} \qquad \boxed{\varepsilon = cT\dot\varphi_r}$$

quartic action (constraints imposed)

Results: Effective formalism for hydrodynamic fluctuations

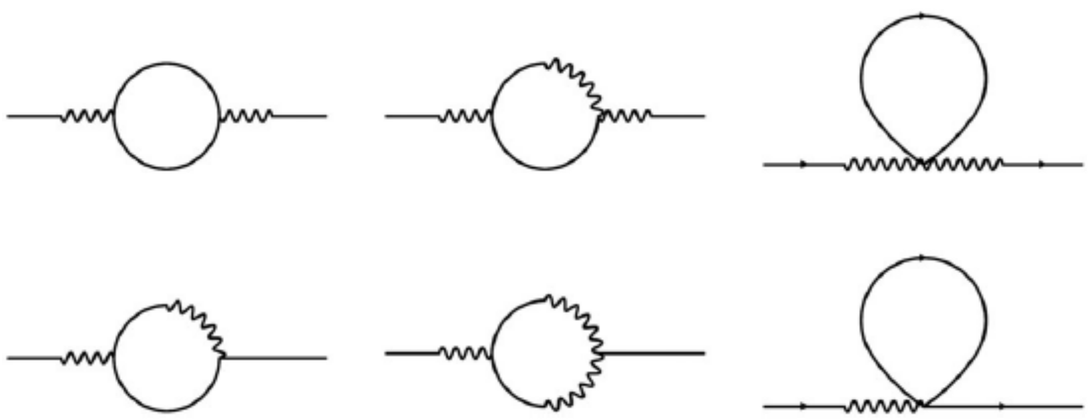
[Chen-Lin, Delacrétaz, Hartnoll; PRL (2019)]

Energy correlator:

$$G^R_{\varepsilon\varepsilon}(\omega,k) = \frac{i[\kappa + \delta\kappa(\omega,k)]Tk^2}{\omega + iDk^2 + \Sigma(\omega,k)}$$

$$\delta\kappa(\omega,k) = \delta\kappa + \kappa_\star(\omega,k),$$
$$\Sigma(\omega,k) = i\delta Dk^2 + \Sigma_\star(\omega,k),$$

$$\mathcal{L} = iT^2\kappa(\nabla\varphi_a)^2 - \varphi_a(\dot\varepsilon - D\nabla^2\varepsilon) + \nabla^2\varphi_a\left(\frac{\lambda}{2}\varepsilon^2 + \frac{\lambda'}{3}\varepsilon^3\right) + icT^2(\nabla\varphi_a)^2(\tilde\lambda\varepsilon + \tilde\lambda'\varepsilon^2) + \cdots,$$

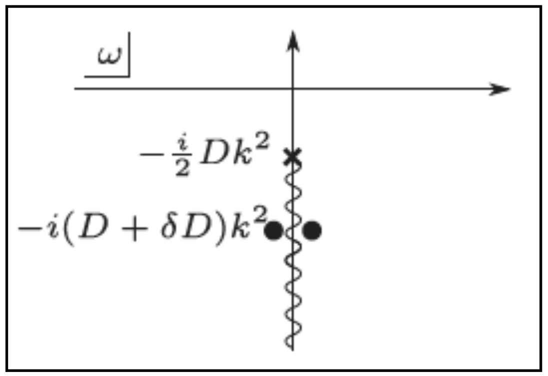


Analytic corrections to transport:

$$\frac{\delta\kappa}{\kappa} = \frac{f_d}{c\ell^d_{\text{th}}}\lambda_\kappa, \qquad \frac{\delta D}{D} = \frac{f_d}{c\ell^d_{\text{th}}}\lambda_D$$

Nonanalytic corrections:

$$\kappa_\star(\omega,k) = f_\kappa(\omega,k)\alpha_d(\omega,k),$$
$$\Sigma_\star(\omega,k) = k^2 f_\Sigma(\omega,k)\alpha_d(\omega,k),$$



$$\alpha_1(\omega,k) = \frac{1}{4}\left(k^2 - \frac{2i\omega}{D}\right)^{-1/2}, \qquad (d=1)$$
$$\alpha_2(\omega,k) = -\frac{1}{16\pi}\log\left(k^2 - \frac{2i\omega}{D}\right), \qquad (d=2)$$
$$\alpha_3(\omega,k) = -\frac{1}{32\pi}\left(k^2 - \frac{2i\omega}{D}\right)^{1/2}. \qquad (d=3)$$

$$f_\kappa(\omega,k) = \frac{cT^2}{D^2}k^2\lambda\tilde\lambda,$$
$$f_\Sigma(\omega,k) = \frac{cT^2}{D^2}[\omega\lambda(\lambda+\tilde\lambda) + iDk^2\lambda\tilde\lambda]$$

nonanaliticities in energy correlator introduce <u>branch point</u> half-way to <u>splitted</u> diffusion pole