Perturbation theory for four-body systems near unitarity

Sebastian König

INT 24-3: Quantum Few- and Many-Body Systems in Universal Regimes

October 21, 2024

NC STATE UNIVERSITY

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Thanks...

...to my collaborators...

- **Feng Wu (IJCLab Orsay)**, **Xincheng Lin (NCSU)**
- U. van Kolck (IJCLab Orsay, U. Arizona, ECT* Trento)
- B. Long, R. Peng (Sichuan U.), S. Lyu (INFN Naples)
- G. Hupin (IJCLab Orsay), K. Kravvaris (LLNL)
- H.-W. Hammer TU Darmstadt), H. Griesshammer (George Washington U.)

...for support, funding, and computing time...

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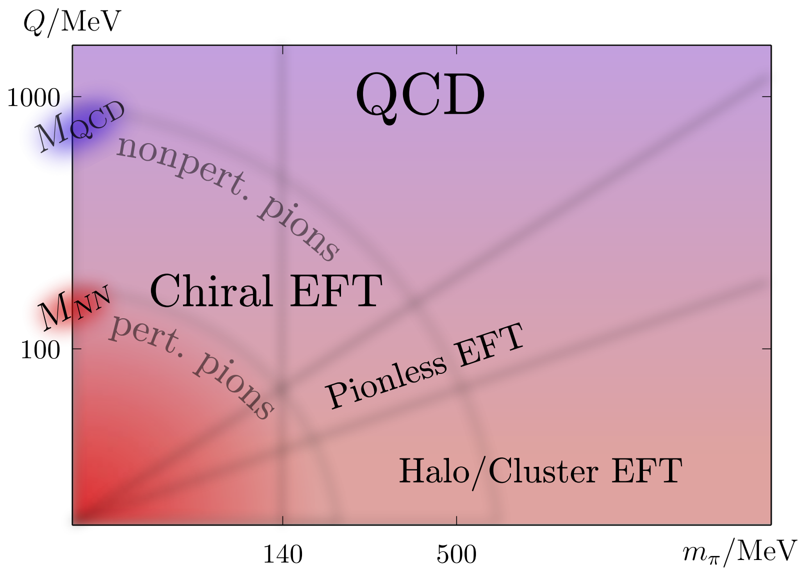
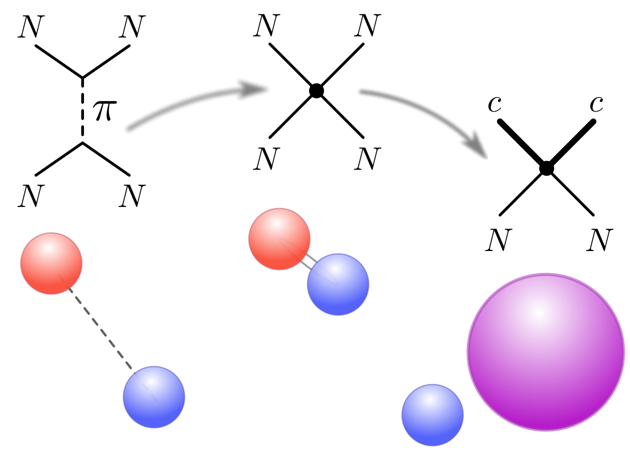
- NCSU HPC Services, Jülich Supercomputing Center

Nuclear effective field theories

- choose **degrees of freedom** approriate to energy scale
- only restricted by **symmetry**, ordered by **power counting**

Hammer, SK, van Kolck, RMP **92** 025004 (2020)



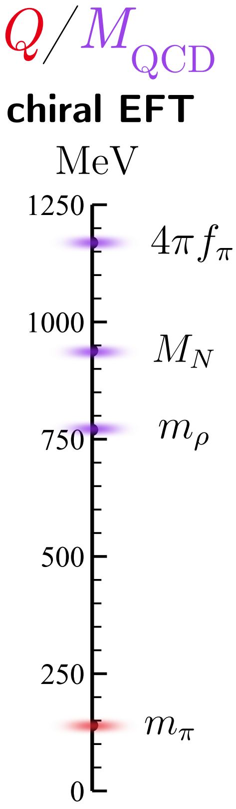


- degrees of freedom here: nucleons (and/or clusters thereof)
- even more effective d.o.f.: rotations, vibrations

Papenbrock, NPA **852** 36 (2011); ...

- **most effective theory depends on energy scale (and nucleus) of interest**

Nuclear scales



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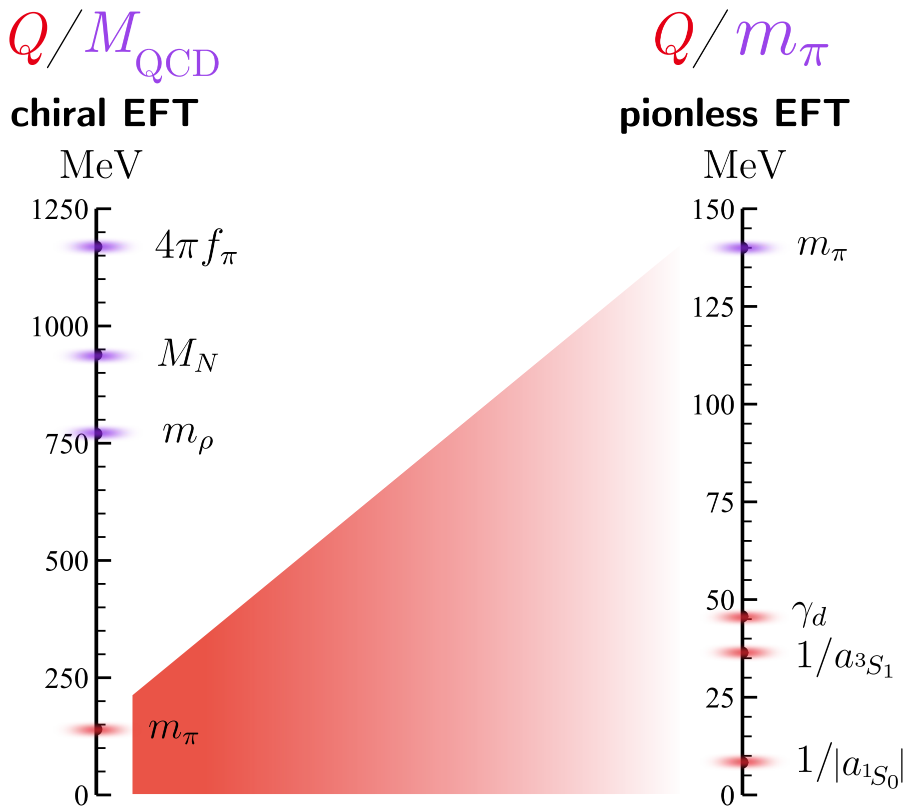

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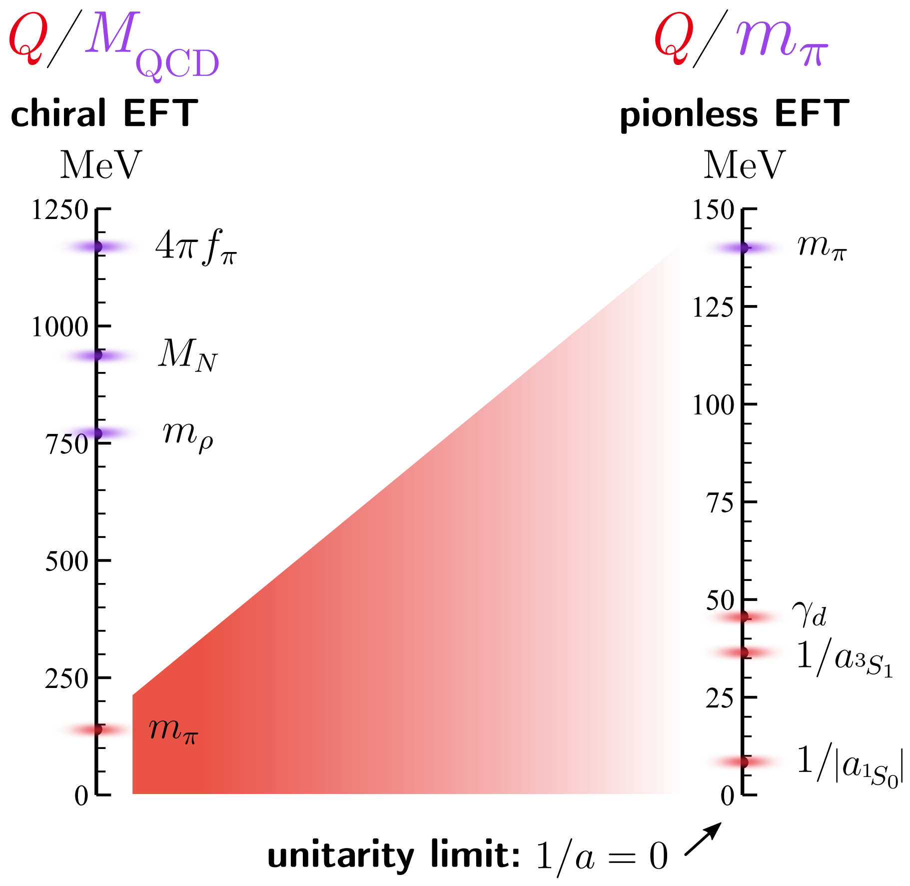
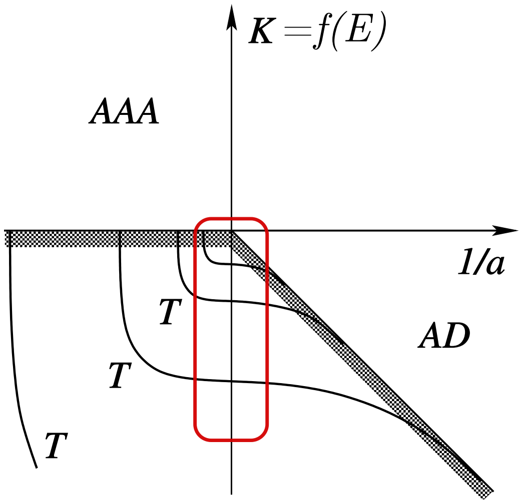


Efimov trimers and tetramers

- **Efimov effect:** infinite tower of three-body states in unitarity limit
- realized experimentally in cold atomic systems Efimov, PLB **33** 563 (1970)
  - scattering length can be tuned via Feshbach resonances



- two-body system is **scale invariant** at unitarity
- three-body scale arises via **dimensional transmutation**
- three-body bound-state energies are **spaced geometrically**
  - $E^{(n+1)} = E^{(n)}/(22.7)^2$

Braaten+Hammer, Phys. Rept. **428** 259 (2006)

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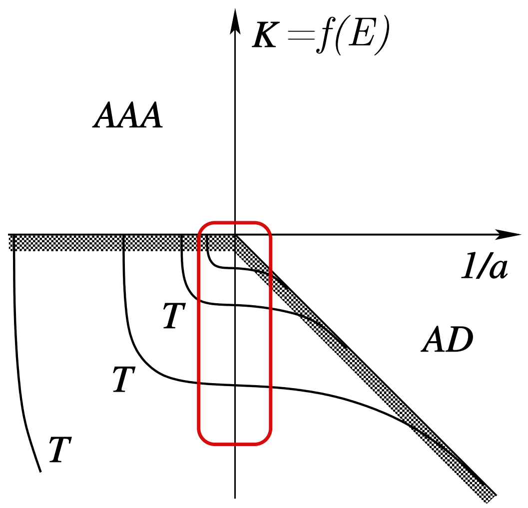

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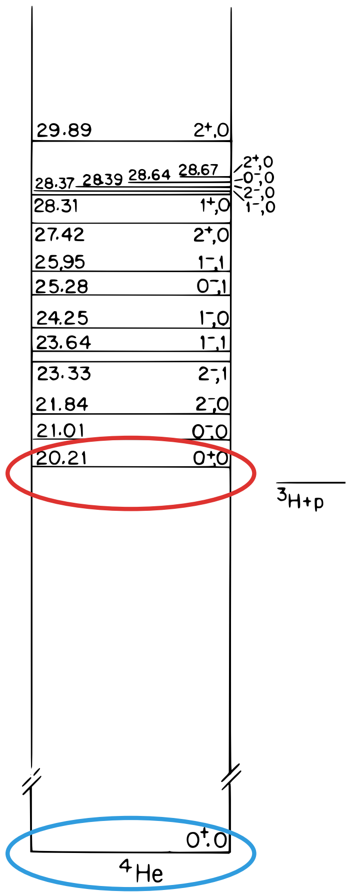
Braaten+Hammer, Phys. Rept. **428** 259 (2006)

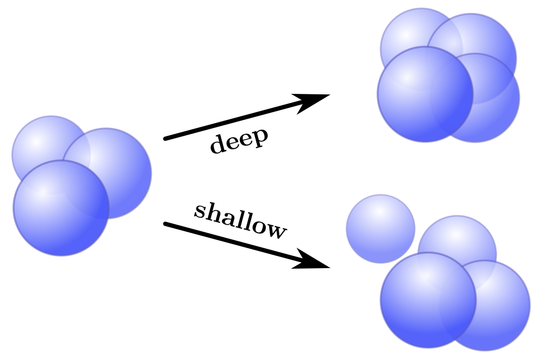
- **each state comes with two associated tetramers**
- plus higher-body clusters beyond that

Hammer+Platter, EPJA **32** 13 (2007); von Stecher, JPB **43** 101002 (2010); ...

Efimov trimers and tetramers



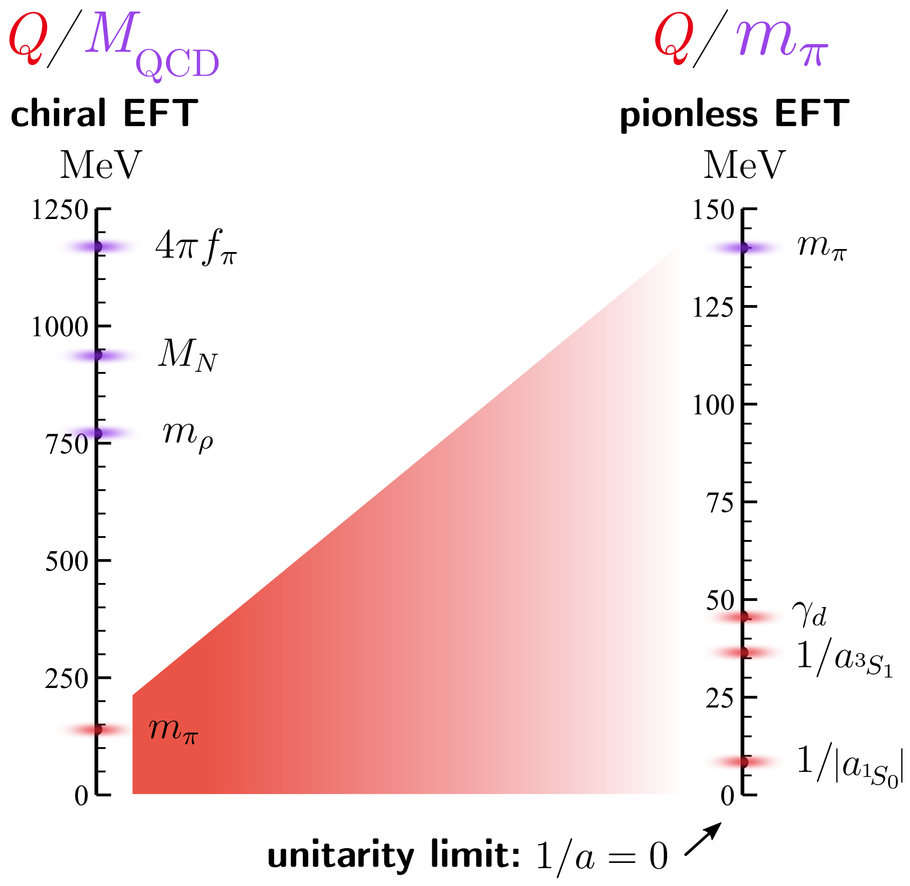
TUNL nuclear data



- **at unitarity**
  - $B_4/B_3 \simeq$ 4.611, $B_{4*}/B_3 \simeq$ 1.002

    Hammer+Platter, EPJA **32** 13 (2007); Deltuva, PRA **82** 040701 (2010)
- **in ^{4}He**
  - ground state at $B_\alpha/B_T \simeq$ 3.66
  - resonance at $B_{\alpha*}/B_T \simeq$ 1.05 (where $B_T = 7.72$)

Nuclear scales revisited



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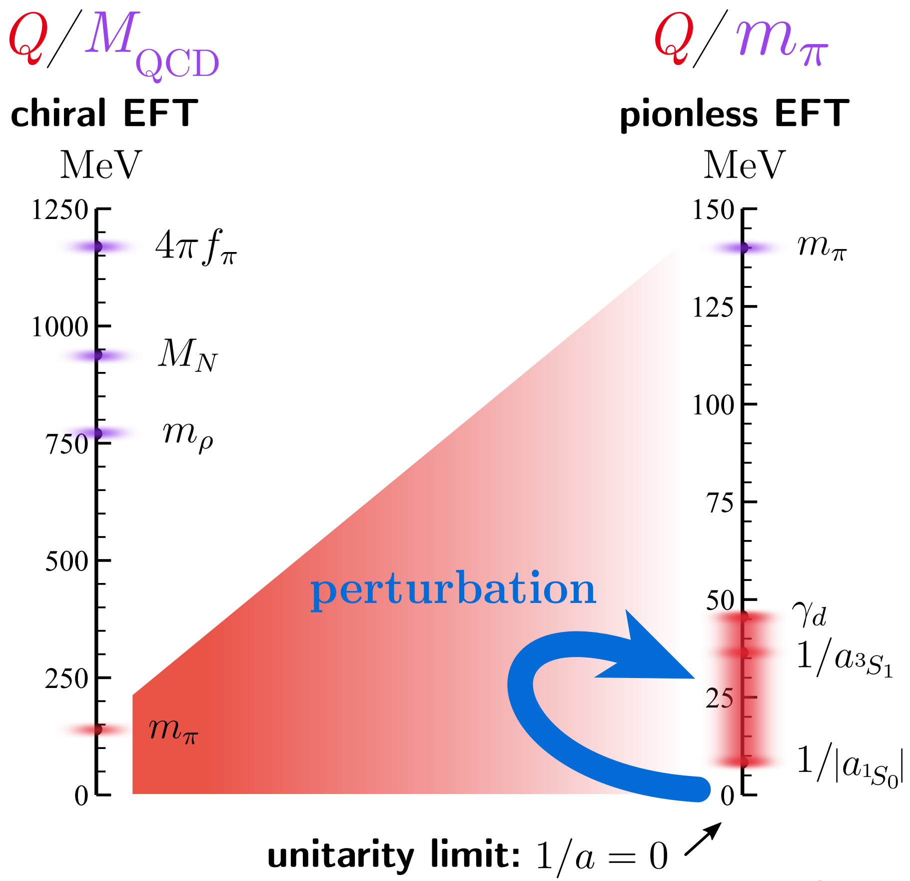

SK et al. PRL **118** 202501 (2017)

The unitarity expansion

Capture gross features at leading order, build up the rest as perturbative "fine structure!"

Nuclear sweet spot

- $1/a < Q_A < 1/R \sim m_\pi$
- $Q_A = \sqrt{2 M_N B_A / A}$

SK et al. PRL **118** 202501 (2017)

A	2	3	4	···	56
$\mathbf{Q_A R}$	0.3	0.5	0.8	···	0.9

$\hookrightarrow$ iron not much different from ^{4}He

van Kolck (2018)

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van Kolck (2018)

- **discrete scale invariance as guiding principle (Efimov effect!)**
  - near equivalence to bosonic clusters, exact SU(4) symmetry

Wigner, Phys. Rev. **51** 106 (1937); Mehen et al., PRL **83** 931 (1999); Bedaque et al., NPA **676** 357 (2000)
Vanasse+Phillips, FB Syst. **58** 26 (2017)
cf. also Kievsky+Gattobigio, EPJ Web Conf. **113** 03001 (2016), ...

Unitarity expansion scheme

SK et al., PRL **118** 202501 (2017)

(1) describe strong force with contact interaction

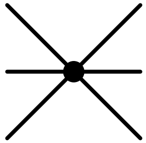
$$C_0 = \underbrace{C_0^{(0)}}_{\text{leading order (LO)}} + C_0^{(1)} + \cdots$$

- momentum cutoff Λ gives "smearing"
- fit $C_0^{(0)}$ to get $a = \infty$ in both NN S-wave channels

(2) fix Efimov spectrum to physical triton energy

- pionless LO three-body force
- **triton as "anchor" at each order**

Bedaque et al., NPA **676** 357 (2000)



(3) include in perturbation theory

- finite a, Coulomb
- range corrections
- all further higher-order corrections

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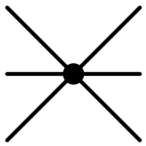


(3) include in perturbation theory

- finite a, Coulomb
- **range corrections** $\leftarrow$ **not actually included so far!**
- all further higher-order corrections

Let's study this for bosonic systems first!

Part I

Bosons

Universal few-boson systems

^{4}He atoms

- ^{4}He atoms are naturally close to the unitarity limit
  - S-wave scattering length $a_2 = 100$ Å, effective range $r_2 = 7.326$ Å much smaller
  - shallow dimer with binding energy ~ 1.3 mK

    Janzen + Azsz, J. Chem. Phys. **103** 9626 (1995)
  - universal trimers and tetramers with binding energies $\mathcal{O}(100)$ mK

Other cold atomic systems

- Feshbach resonances can be used to tune the scattering length in cold atomic gases
- Efimov effect has been observed for a variety of systems by now

Relation to nuclear physics

- in the Wigner SU(4) limit, the equations for the S=1/2 three-nucleon system decouple into two components
- **one of these components is equivalent to the bosonic equation**

Bedaque, Hammer, van Kolck, NPA **676** 357 (1999)

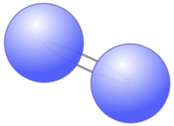
Many talks about all this at this workshop!

U. van Kolck (Monday), H. Griesshammer (Monday), X. Lin (Wednesday), ...

Numerical approach

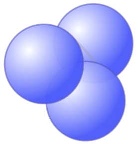
Unified (2-, 3-, 4-body) numerical framework

Two-body system



- separable regulator for contact interactions: $V = C_0 |g\rangle\langle g|$
- can be solved analytically to get scattering amplitudes

Three-body system



- Faddeev equations: $|\psi\rangle = G_0 t P |\psi\rangle + (G_0 + G_0 t G_0) V_3 |\Psi\rangle$
- full wave function: $|\Psi\rangle = (1 + P)|\psi\rangle$
- **used to fit three-body force**

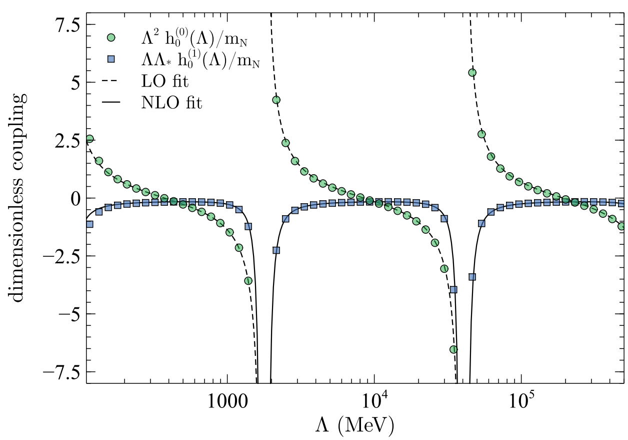
Four-body system

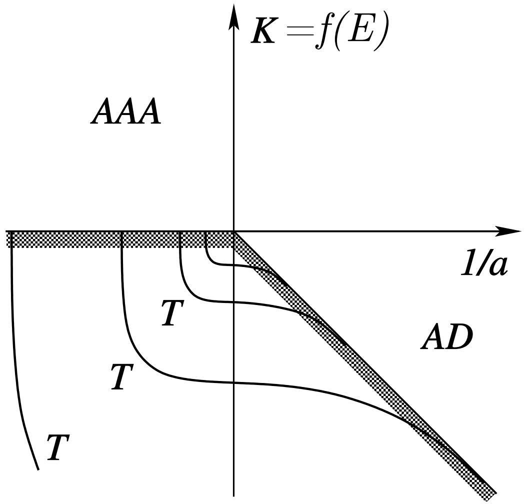
- Faddeev-Yakubowsky equations: two components $|\psi_{A,B}\rangle$
- full wave function involves both components:
  - $|\Psi\rangle = (1 - P_{34} - P P_{34})(1 + P)|\psi_A\rangle + (1 + P)(1 + \tilde{P})|\psi_B\rangle$

Deep trimers

- **the physical Efimov effect is an infrared phenomenon**
  - accumulation point of three-body bound states (trimers) at zero energy
- **however, the EFT may exhibit a parallel UV limit cycle**
  - famous log-periodic cutoff dependence of the three-body force
  - with each pole in the three-body coupling constant, a new deep trimer enters the spectrum

Bedaque, Hammer, van Kolck, PRL **82** 463 (1999); NPA **646** 444 (1999)





- **occurrence depends on regularization scheme**
  - no UV limit cycle observed for local regulators

Kirscher + Gazit, PLB **755** 253 (2016)

Why do we care?

Deep trimer removal

- at some cutoff a physical tetramer is associated with the deepest trimer
- as we increase the cutoff, a yet deeper trimer appears
- the physical tetramer can then decay into that deep trimer plus a particle
  - **i.e., the physical tetramer becomes a resonance**

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gg3po, via Wikimedia Commons (GPL)

Trimer sweeping

- calculate the full wave function $|\Psi_3\rangle$ of the deep trimer
- then replace $V_3 \to V_3 + \lambda|\Psi_3\rangle\langle\Psi_3|$ in the interaction, λ = large number
  - **this removes the deep trimer from the negative-energy spectrum!**
- well-established technique in general, known for decades — Lehman, PRC **25** 3146 (1982)
- used for example to remove spurious two-nucleon states in Chiral EFT — Nogga, Timmermans, van Kolck, PRC **72** 054006 (2005)
- **requires fully generic FY equation (no separable simplification!)**

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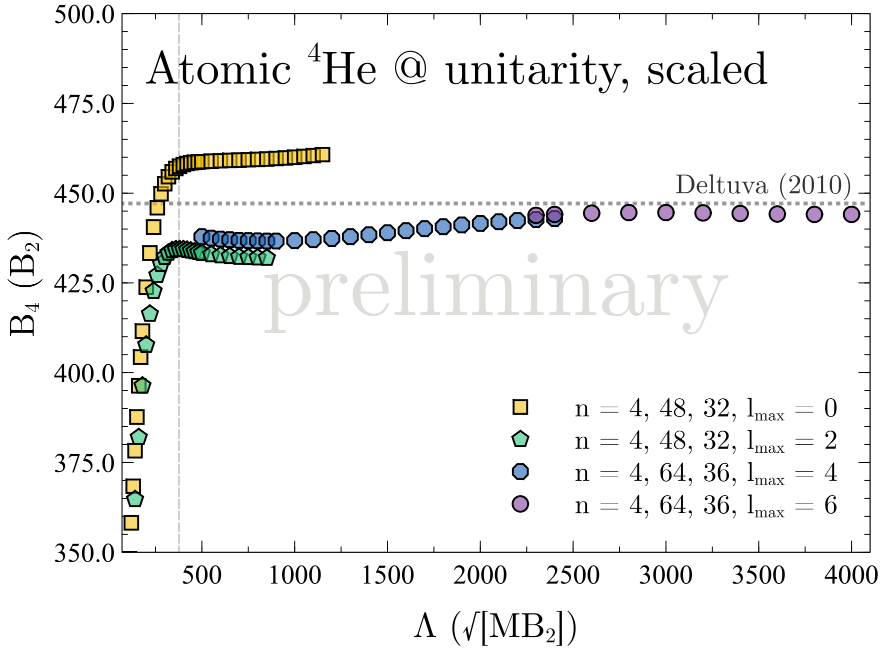
- **requires fully generic FY equation (no separable simplification!)**

Alternative approach

- it is also possible to work with four-body equations that feature a full trimer propagator and remove the deep state directly at that level — Lin, PRC **109** 024002 (2024)

see talk by Xincheng Lin

Four-boson ground-state convergence



- energies expressed in units of $B_2 = 1$ (even though system at unitarity!)
  - would be more sensible to use B_3 as reference energy
- three-body state fixed at $B_3 = 96.971\,B_2$, based on physical ^{4}He atoms
- apparent convergence to expected $B_4^{(0)} = 4.6108\,B_3$ for large cutoffs
  - however, oscillations overlay simple $1/\Lambda$ dependence

Deltuva, PRA **82**, 040701 (2010)

Range corrections

Recap

- **recall the unitarity expansion at next-to-leading order:**
  - Coulomb (not in this talk), finite scattering length ($1/a \neq 0$), range corrections
- in the standard theory, $1/a$ enters at leading order, range correction is NLO
- unitarity scheme pairs the expansions in $1/(Qa)$ and Q/M_{hi}

Separable Hamiltonian formalism

$$H(p,p') = H_0 + C_0\, g(p)g(p') + C_2\, g(p)\left[p^2 + p'^2\right]g(p') + \cdots$$

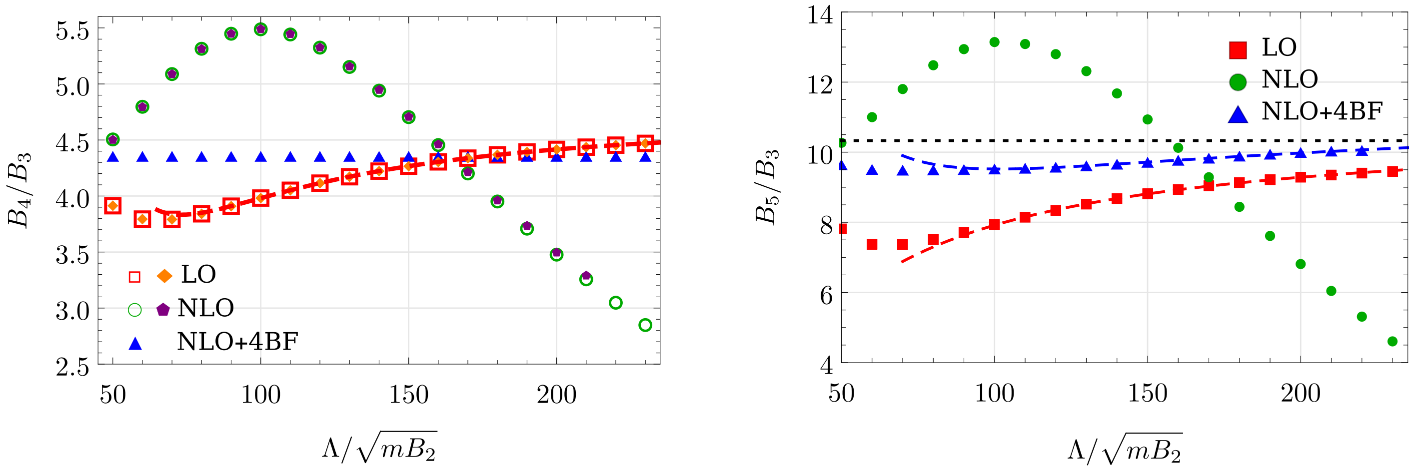
- $g(p) = \exp\left(-(p/\Lambda)^n\right)$ with $n = 2, 4, \cdots$ implements **regularization**
- LECs are split into orders and run with the cutoff Λ
  - $C_0 = C_0^{(0)}(\Lambda) + C_0^{(1)}(\Lambda) + \cdots$, $C_2 = C_2^{(1)}(\Lambda) + \cdots$
  - reproduce effective range expansion: $p \cot\delta(p) = -\dfrac{1}{a} + \dfrac{r^2}{2}p^2 \cdots$
- **Note:** this uses a **momentum-dependent** formulation
- approaches with dimer fields implement range correction via energy dependence

see e.g. talk by X. Lin

NLO four-body force

- full next-to-leading order includes range corrections $\sim C_2^{(1)}\,(p^2 + p'^2)$
- four-boson energy does not converge with cutoff
- **promotion of four-body force to NLO**

Bazak, Kirscher, SK et al., PRL **122** 143001 (2019)



- **inclusion of four-body force stabilized five- and six-body system as well**
- general prediction for promotion of many-body forces (for bosons!)
- only need a single four-body datum, rest remains prediction at NLO
- higher-body few-body forces likely promoted as well (but not for nucleons)

Consequences

Four-body ground-state energy becomes input at NLO...

..but other observables remain predictions, e.g.:

- tetramer excited state talk by X. Lin
- ground-state radius
- other static properties

Perturbative Faddeev scheme

SK, EPJA **56** 113 (2020); cf. Vanasse, PRC **88** 044001 (2013)

Basic setup

- full wavefunction: $|\Psi\rangle = (1 + P)|\psi\rangle$
- perturbative expansion: $|\Psi\rangle = |\Psi^{(0)}\rangle + |\Psi^{(1)}\rangle + \cdots$
- Faddeev equation: $|\psi^{(0)}\rangle = K^{(0)}|\psi^{(0)}\rangle$ with $K^{(0)} = G_0 t^{(0)} P$

$\rightsquigarrow$ NLO energy shift $B^{(1)} = \langle\Psi^{(0)}|V^{(1)}|\Psi^{(0)}\rangle$

Wave-function correction

- set $K^{(1)} = B^{(1)}(G_0 + G_0 t^{(0)} G_0) + G_0 t^{(1)} P$
- then $(\mathbf{1} - K^{(0)})|\psi^{(1)}\rangle = K^{(1)}|\psi^{(0)}\rangle$
  - singular part from LO solution projected out
- **need only solve linear system with same kernel as LO equation!**

$\rightsquigarrow$ N2LO energy corrections $B^{(2)} = \langle\Psi^{(1)}|V^{(1)}|\Psi^{(0)}\rangle + \langle\Psi^{(0)}|V^{(2)}|\Psi^{(0)}\rangle$

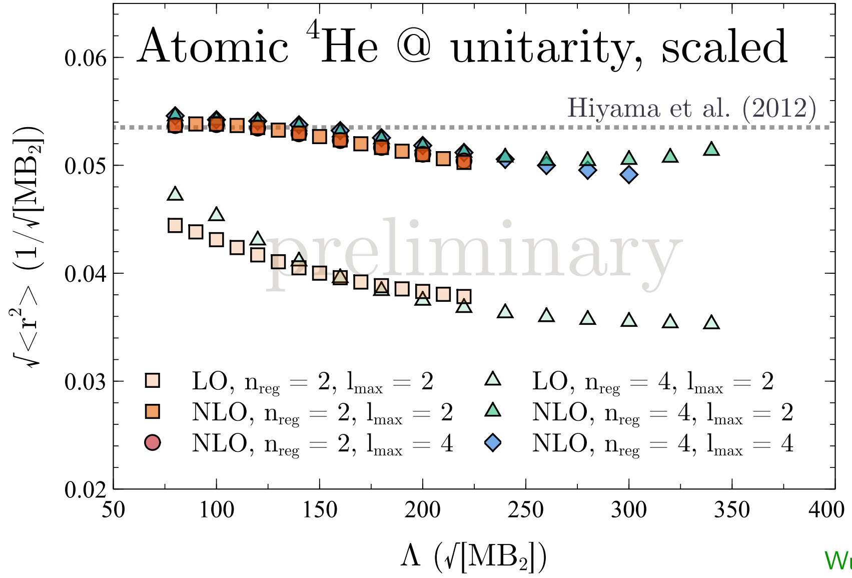
$\rightsquigarrow$ **NLO corrections for operators (form factor $\rightarrow$ radius)**

- **works essentially the same way for Faddeev-Yakubowsky equations**

Four-boson ground-state radius

- calculations orchestrated and executed by Feng Wu
- radius calculated from form factor in strict perturbation theory
  - $F_C(q) = \langle\Psi|\rho(q)|\Psi\rangle \rightsquigarrow \langle r^2\rangle = -\frac{1}{6}\frac{d^2}{dq^2}F_C(q)\Big|_{q\to 0} = \langle r^2\rangle^{(0)} + \langle r^2\rangle^{(1)} + \cdots$

SK, EPJA **56** 113 (2020)



Wu, SK, et al., work in progress

- jobs used 48 ($l_{max} = 2$) and 64 ($l_{max} = 4$) momentum mesh points
- **NLO range corrections shift radius towards potential-model result**

Hiyama + Kamiura, PRA **85**, 022502 (2012)

Part II

Nucleons

Recap

- 3N/4N systems dominated by "bosonic" component
- ^{4}He binding energy becomes input at NLO
- ^{4}He radius can still be predicted
- **also interesting:** excited state in ^{4}He, more nucleons

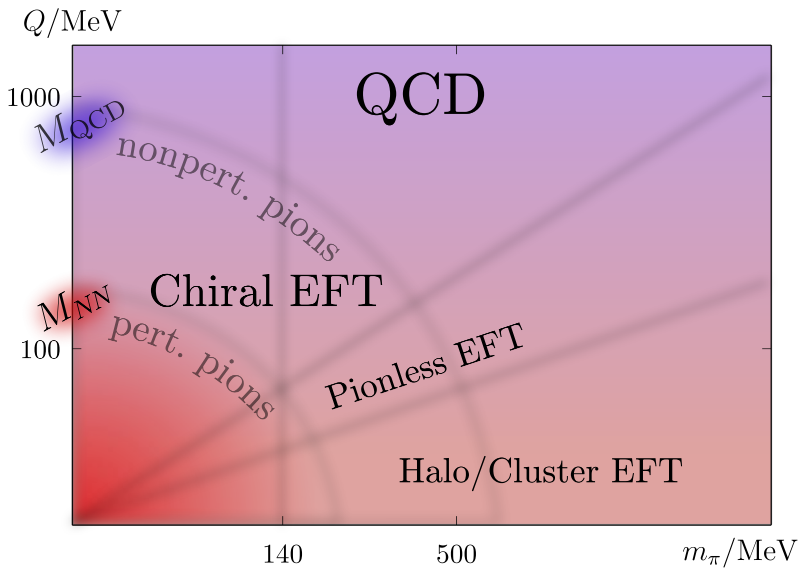
Hupin, SK, Kravvaris, van Kolck, Wu, work in progress

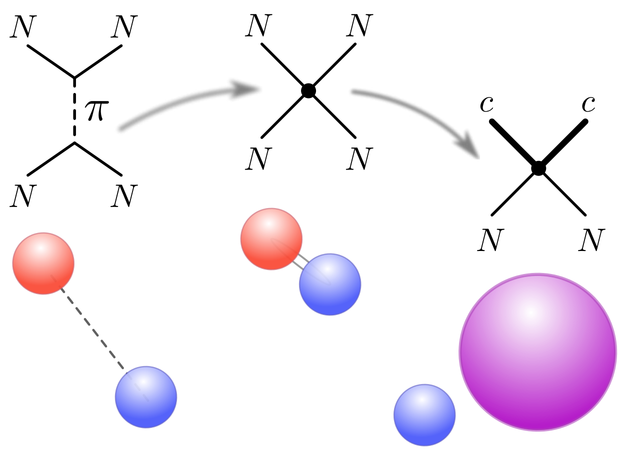
But let's talk about something else for a moment...

Nuclear effective field theories

- choose **degrees of freedom** approriate to energy scale
- only restricted by **symmetry**, ordered by **power counting**

Hammer, SK, van Kolck, RMP **92** 025004 (2020)





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- even more effective d.o.f.: rotations, vibrations

Papenbrock, NPA **852** 36 (2011); ...

- **most effective theory depends on energy scale (and nucleus) of interest**

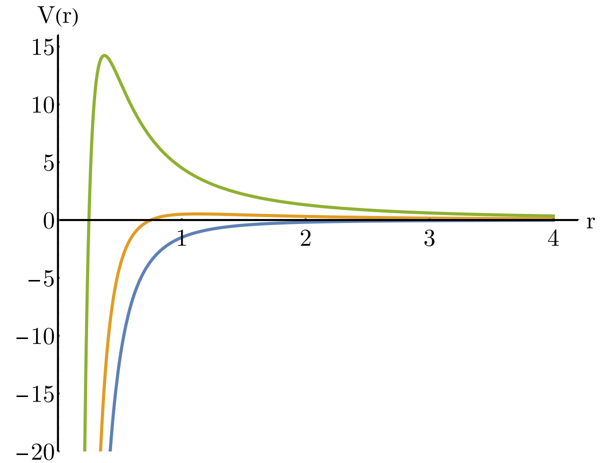
Perturbative pions?

- consider the effective total potential:

$$V(r) = V_{\text{OPE}}(r) + \frac{L(L+1)}{r^2}$$

- for $L > 0$, this "shields" the nucleons from the singular attraction of $V_{\text{OPE}}(r)$
- critical momentum characterizes perturbativeness

Birse, PRC 2006



Partly perturbative pions

- observation is the basis for perturbatively renormalized chiral EFTs (not this talk!)

Nogga, Timmermans, van Kolck, PRC 2005

KSW counting

- basically, **expand nuclear force around pionless EFT** — Kaplan et al., PLB/NPB 1998
- compelling idea, alas: poor convergence properties — Cohen+Hansen 1999; Fleming et al., NPA 2000
- recent work: problematic only in low partial waves! — Wu + Long 2019; Kaplan, PRC 2020

Perturbative pions!

Observation 1

- KSW scheme works for two-nucleon partial waves with $l > 0$, except $^3P_0$
  - for center-of-pass momenta below the Δ threshold

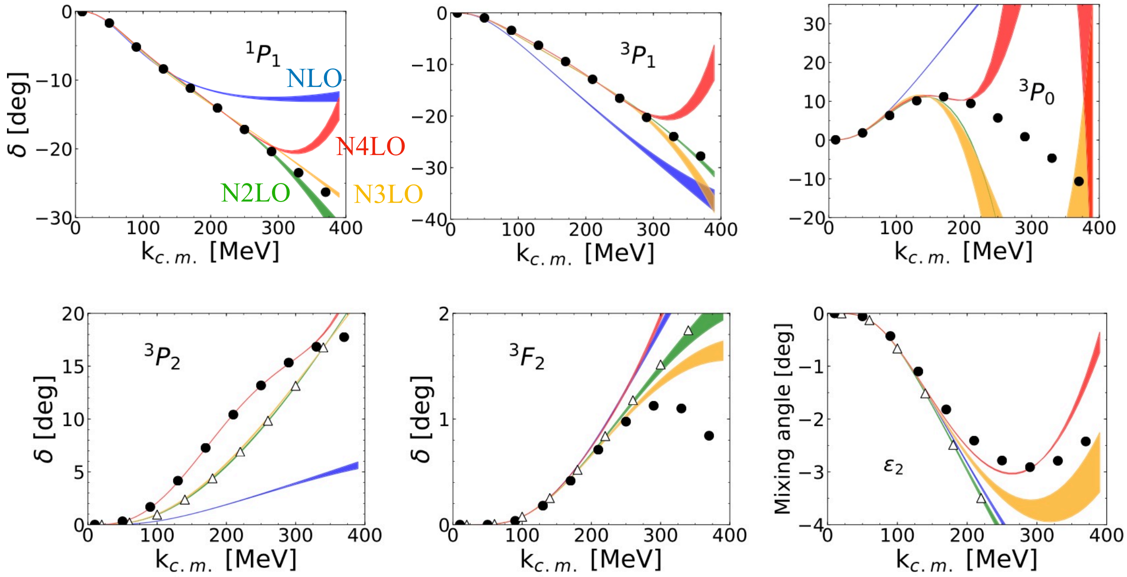
Wu + Long (2019); Kaplan, PRC (2020)

- however, Born approximation with OPE + contact works for $^3P_0$ scattering
  - **$^3P_0$ converges with a P-wave contact promoted to NLO!**

Peng, Lyu, Long (2020)

Two-nucleon phase shifts

- **good convergence of two-nucleon P-waves** (except $^3P_0$)
  - breaks down at $k \sim 300$ MeV (Δ production threshold)
- similar picture for yet higher partial waves



Wu + Long, PRC **99** 024003 (2019)

B. Long, talk at Chiral Dynamics 2024

SAID: gwdac.phys.gwu.edu

- circles: SAID data, triangles: OPE + once iterated

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Peng, Lyu, Long (2020)

Observation 2

- ${}^3S_1$-${}^3D_1$ mixing angle vanishes in the combined unitarity+chiral limit
  - $1/a_{{}^3S_1} \to 0$, $m_\pi \to 0$ implies $\epsilon \to 0$
  - this happens despite the tensor force still being strong in the chiral limit
  - however, this is an on-shell only effect

Lyu, Zuo, Peng, SK + Long, in preparation

- at N2LO, there is still a large correction to the mixing angle (and ${}^3D_1$ phase shift)
  - **this can be fixed by promoting just the SD mixing term to NLO!**
- related recent work in similar direction

talk by Harald last week Teng + Griesshammer, 2410.09653 [nucl-th]

Perturbative pions!

Observation 1

- KSW scheme works for two-nucleon partial waves with $l > 0$, except $^3P_0$
  - for center-of-pass momenta below the Δ threshold

Wu + Long (2019); Kaplan, PRC (2020)

- however, Born approximation with OPE + contact works for $^3P_0$ scattering
  - **$^3P_0$ converges with a P-wave contact promoted to NLO!**

Peng, Lyu, Long (2020)

Observation 2

- $^3S_1$-$^3D_1$ mixing angle vanishes in the combined unitarity+chiral limit
  - $1/a_{^3S_1} \to 0$, $m_\pi \to 0$ implies $\epsilon \to 0$
  - this happens despite the tensor force still being strong in the chiral limit
  - however, this is an on-shell only effect

Lyu, Zuo, Peng, SK + Long, in preparation

- at N2LO, there is still a large correction to the mixing angle (and $^3D_1$ phase shift)
  - **this can be fixed by promoting just the SD mixing term to NLO!**
- related recent work in similar direction

talk by Harald last week Teng + Griesshammer, 2410.09653 [nucl-th]

Bottom line

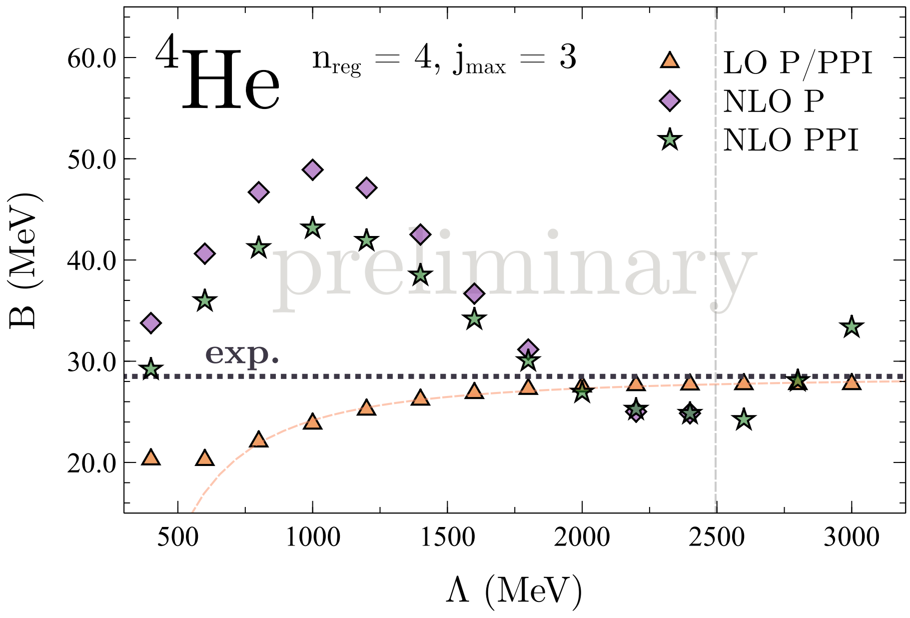
- **together, these two modifications fix perturbative pions for 2N scattering**

Finally

Let's look at some few-nucleon results!

^{4}He energy at NLO

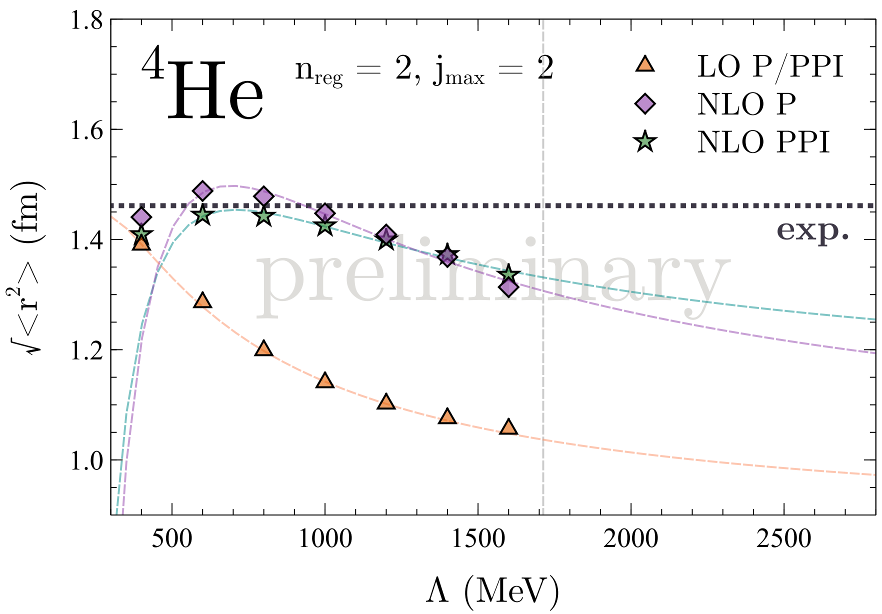
- consider first a calculation without four-nucleon force at NLO
- calculation with 48 momentum mesh points
  - including some large cutoffs with deep triton state removed



- **do perturbative pions eliminate the need for a four-nucleon force?**
  - it does not look like they do

^{4}He radius at NLO

- now we include the four-nucleon force and calculate the radius



Remarks

- calculations now with 32 momentum mesh points
- dashes lines fit a polynomial in $1/\Lambda$
  - standard approach, but may not capture actual Λ dependence!

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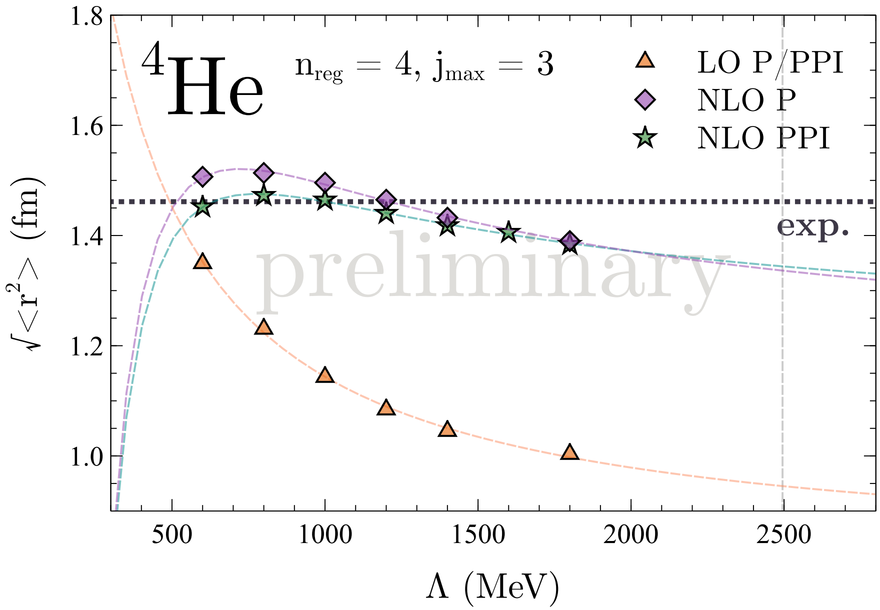
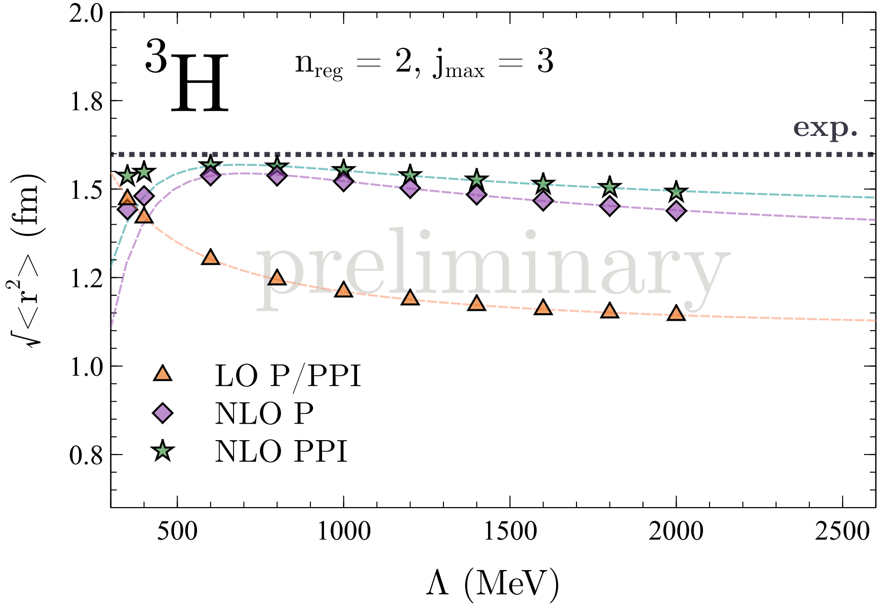


Remarks

- calculations now with 32 momentum mesh points
- dashes lines fit a polynomial in $1/\Lambda$
  - standard approach, but may not capture actual Λ dependence!
- no clear improvement of radius from pions

^{3}H radius at NLO

- the triton energy is used to fix the three-body force...
- ...but of course we can predict the triton radius!



Remarks

- this calculations is again with 48 momentum mesh points
- range correction studied previously in Pionless EFT
- somewhat more significant impact of pions on the radius

Vanasse, PRC **95** 024002 (2017)

Summary and outlook

Bosons

- removal of deep trimers enables large-cutoff calculations
- tetramer ground state energy converges to known universal value
- good convergence of four-boson radius at NLO

Nucleons

- range corrections shift ^{3}H and ^{4}He radii towards experiment
- small modifications enable perturbative-pion expansion
- four-nucleon at NLO still needed with perturbative pions
- perturbative pions have only minor effects on radii

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Outlook

- comprehensive study of tetramer excited state
- full nuclear unitarity expansion at NLO
- push four-nucleon calculations to larger cutoffs

Thanks...

...to my collaborators...

- **Feng Wu (IJCLab Orsay)**, **Xincheng Lin (NCSU)**
- U. van Kolck (IJCLab Orsay, U. Arizona, ECT* Trento)
- B. Long, R. Peng (Sichuan U.), S. Lyu (INFN Naples)
- G. Hupin (IJCLab Orsay), K. Kravvaris (LLNL)
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...and to you, for your attention!