Four-body systems at large cutoffs in short-range EFTs up to NLO

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Background: Goals and Key ingredients

Semi-analytical EFT studies on four-body (4B) systems

- QCD $\xrightarrow[\Lambda_\chi \sim \mathcal{O}(1)\text{GeV}]{}$ ChEFT $\xrightarrow[\Lambda_{NN} \sim 300\text{MeV}]{}$ KSW $\xrightarrow[\Lambda_{\not{\pi}} \sim 140\text{MeV}]{}$ $\not{\pi}$**EFT**
- Integral equations, e.g., Faddeev, diagrammatic
- cold ^{4}He atoms, ^{4}He nuclei, nuclear cluster/halo systems, etc

Background: Goals and Key ingredients

Renormalization group (RG) in few-body systems, e.g.,

- 2 body: Manifest power counting using dim. reg. with power divergence subtraction
- 3 body: three-body force needed at LO, Efimov physics, and discrete scaling symmetry
- 4 body: four-body force needed starting at NLO (to be shown at *large* cutoffs)

Universality in 4B systems

- Universal relations in the unitarity
- Expansion around unitarity
- Relevant for four-nucleon systems (Wigner-SU(4) symmetry)

Contents

- **Intro**: Contact EFT; 2B, 3B and 4B integral equations
- **Problem**: deep 3B poles make large-cutoff 4B calculations difficult
- **Method**: Addressing 3B poles
- **Results at LO and NLO**: 4B binding energies of cold ^{4}He atoms, their cutoff dependence, and 4B force
- **Universal relations and Comparing with FY results**

Two-Body Systems

2B System with a Pole

$$\frac{m}{2\pi}A_2 = \frac{1}{-\frac{1}{a} + \frac{1}{2}rp^2 + \cdots - ip} \text{ (effective range expansion)} \tag{1}$$

$$= \frac{1}{-\frac{1}{a} - ip}\left[1 + \underbrace{\left(\frac{rp}{2}\right)\left(\frac{ap}{1+iap}\right)}_{Q} + \cdots\right], \text{ if } |ap| \sim 1, |ra^{-1}| \sim Q \ll 1$$

2B pole (non-perturbative) reproduced at the lowest order

2B System with a Pole

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Relevant 2B systems with $|a/r| \gg 1$?

$$\text{(2N) } {}^3S_1 \text{ deuteron} : a_t \approx 5.4 \text{ fm}, \; r_t \approx 1.7 \text{ fm}$$

$$\text{(2N) } {}^1S_0 \text{ virtual state} : a_s \approx -24 \text{ fm}, \; r_s \approx 2.7 \text{ fm}$$

$$\text{(2 Boson) Cold atomic } {}^4\text{He} : a_{^4\text{He}} \approx 100 \text{ Å}, \; r_v \approx 7 \text{ Å}$$

2B System with a Pole

$$iA = \frac{4\pi}{m_N}\frac{i}{k\cot(\delta) - ik} \overset{ERE}{=} \frac{4\pi}{m_N}\frac{i}{-\frac{1}{a} + \frac{rk^2}{2} + \cdots - ik}$$

$$\approx \frac{4\pi}{m_N}\frac{i}{-\frac{1}{a} - ik} =$$

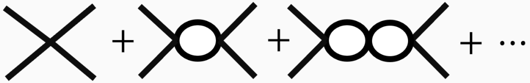

Figure 1: Reproduce 2B pole using Born series with 2B contact interaction (geometric sum)

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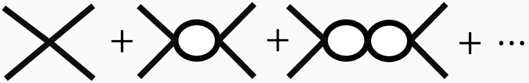

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Why sum to all orders? How to treat different interactions?

Some Concepts in EFT

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- **Renormalization (cutoff dependence) can inform power counting**

Three-Body Systems

3B System in Contact EFT

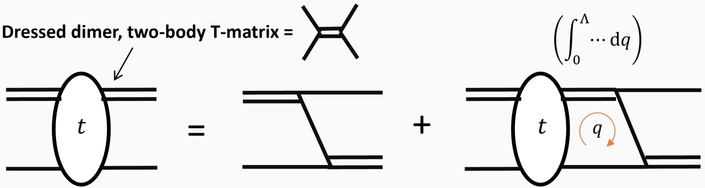


Figure 2: 3B integral equation, equivalent to Faddeev equation

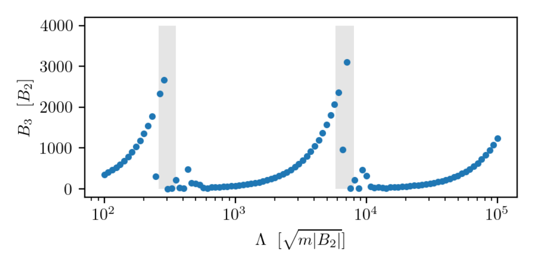
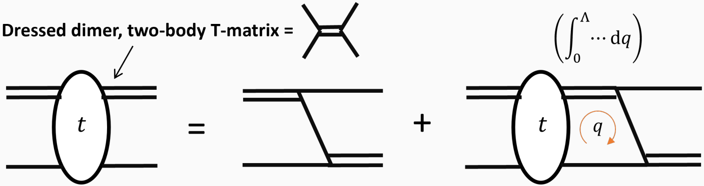


Figure 3: 3B binding energies with 2B contact interaction only

3B System in Contact EFT



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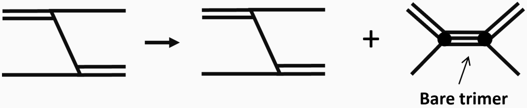


Figure 5: Promoting 3B force to LO to absorb cutoff dependence*

- **LO contact EFT: 2B and 3B force (non-perturbative)**

*Bedaque, Hammer, and Kolck 1999

3B System in Contact EFT

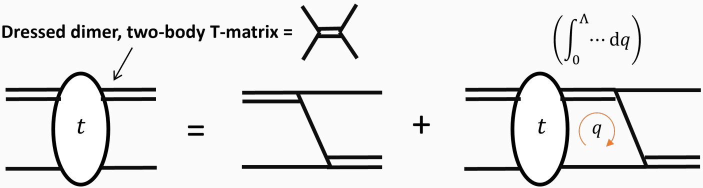
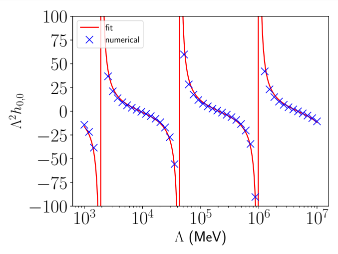
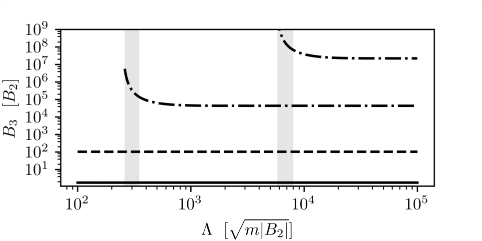


Figure 6: 3B integral equation, equivalent to Faddeev equation



(a) Cutoff dependence of 3B force (illustrative only, ignore units)



(b) Binding energies of Efimov states as a function of cutoff

2B and 3B Systems in Short

Recap of Key concepts in 2B and 3B systems:

- LO interactions: 2B and 3B contact (zero-derivative)
- 2B and 3B bound states occur at LO
- 3B interaction required at LO for RG invariance

Four-Body Systems at LO

4B integral equation

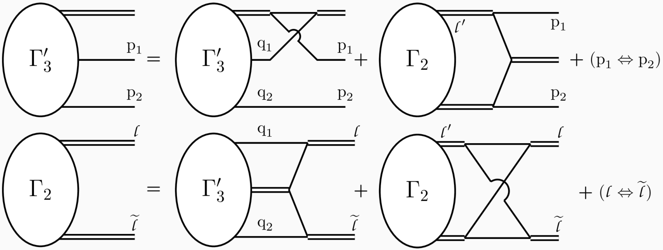


Figure 8: 4B integral equation in terms of 2B force *

- Two types fragmentation: (3B+1B) and (2B+2B)
- 3B poles occurs from iteration of (3B+1B) diagrams

*Brodsky et al. 2006

4B integral equation

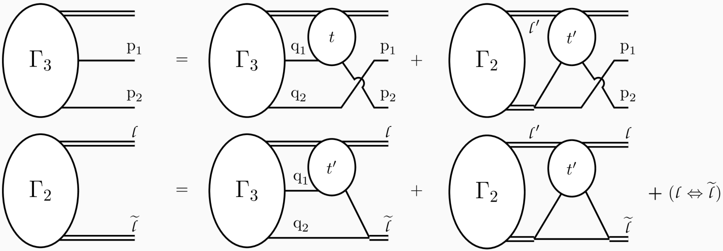
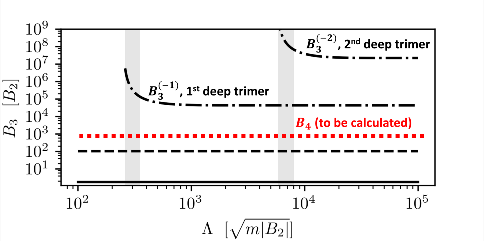


Figure 9: 4B integral equation using 3B amplitude *

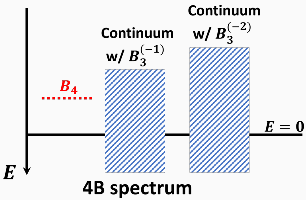
- Can **directly address trimer poles** of 3B amplitude. **But why do we care?**

*Lin 2024

Why worry about 3B poles?



(a) (Schematic) Location of 4B binding energy



(b) 4B “bound” state and continuum states associated with deep trimers

- 3B+1B decay channel available at large cutoffs
- 4B bound state become unstable (become resonance) at large cutoffs
- **Problem: Deep trimers lead to difficult on assessing cutoff dependence**

Dealing with 3B poles in 4B equation

&

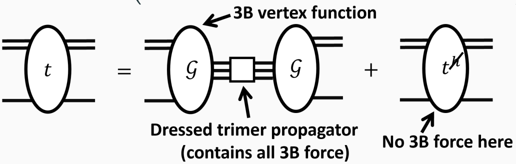
LO Results (for atomic ^4He)

Dealing with trimer poles (Method A)

Method A:* **isolate the singularity and then include/exclude it** by Cauchy principal value prescription

$$\hat{G}_3 = \frac{|\psi\rangle R_\psi \langle\psi|}{E + B_3} + \cdots$$

$$= |\psi\rangle R_\psi \langle\psi| \left(\pm i\pi\delta(E + B_3) + \text{p.v.}\left(\frac{1}{E + B_3}\right)\right) + \cdots$$



- **Large-cutoff calculations now possible with 4B equation in terms of 3B amplitudes**
- **Can compute 4B decay width perturbatively**

*Lin 2024

Results: $B_4^{(n)}$ at LO (cold ^{4}He atoms)

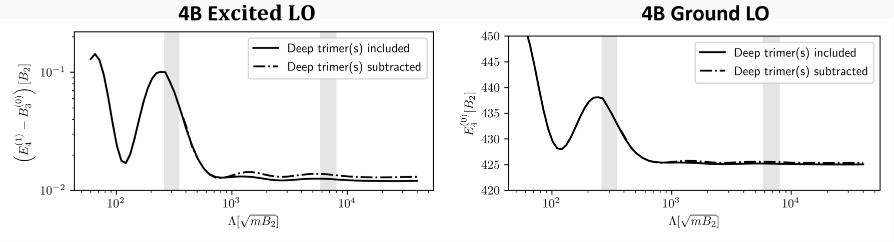


Figure 11: Convergence of $B_4^{(n)}$ at LO

- Gray bands indicate occurance of deep trimer poles
- Convergence $\Rightarrow$ No 4B force needed at LO.*
- Large cutoffs (regulator-dependent) needed for convergence

*See Platter, Hammer, and Meissner 2004 at relatively low cutoffs

Results: $B_4^{(n)}$ at LO (cold ^{4}He atoms)

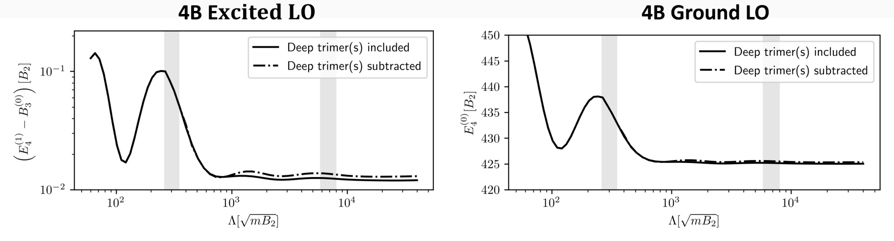


Figure 12: Convergence of $B_4^{(n)}$ at LO

	$B_3^{(0)}$ [mK]	$E_4^{(1)}$ [mK]	$E_4^{(0)}$ [mK]
This work	***128.500**	**128.517(1)**	**526.1(5)**
Platter et al. 2004	127	128[3]	492[25]
H&K 2012	126.4	127.33	558.98

Table 1: Results for cold ^{4}He atoms. First two rows are LO EFT calculations. Hiyama and Kamimura 2012 uses realistic potential.

Four-Body System at NLO

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Preliminary Results (for atomic ^{4}He)

Results: $B_4^{(n)}$ at NLO (cold ^{4}He atoms)

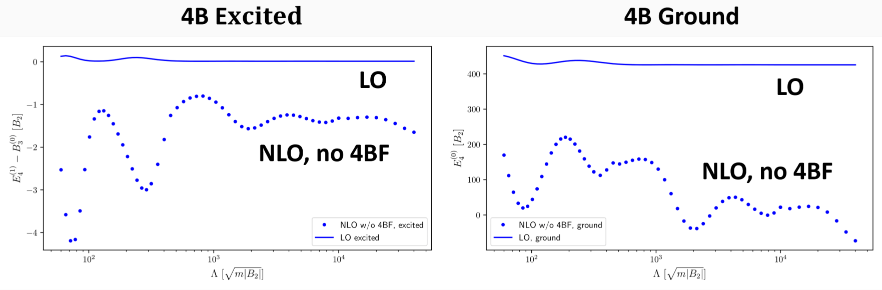
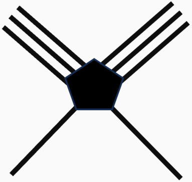


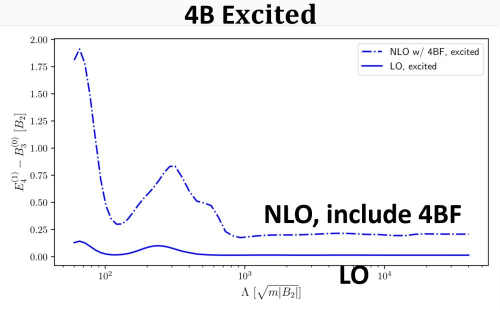
Figure 13: Cutoff dependence of $B_4^{(n)}$ at NLO without 4B force

- **Strong cutoff dependence if no 4B force**

Results: 4B force and $B_4^{(1)}$ at NLO (cold ^{4}He atoms)



(a) 4B force

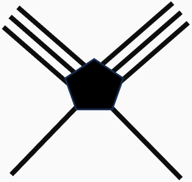


(b) $B_4^{(1)}$ w/ 4B force fitted to $B_4^{(0)}$

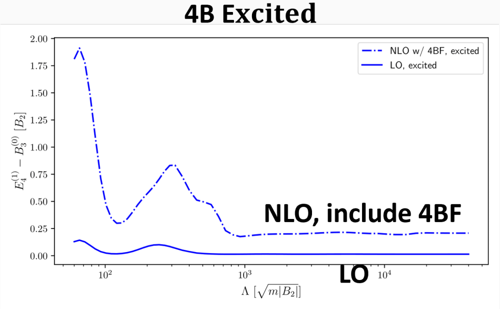
- **4B force needed at NLO for renormalization group invariance***

*See Bazak et al. 2019 at low cutoffs

Results: 4B force and $B_4^{(1)}$ at NLO (cold ^{4}He atoms)



(a) 4B force



(b) $B_4^{(1)}$ w/ 4B force fitted to $B_4^{(0)}$

	$B_3^{(0)}$ [mK]	$E_4^{(1)}$ [mK]	$E_4^{(0)}$ [mK]
This work NLO	***128.500**	**128.83(1?)**	***557**
H&K 2012	126.4	127.33	558.98

Table 2: Results for cold ^{4}He atoms. First row is NLO EFT calculation. Hiyama and Kamimura 2012 uses realistic potential.

4B Universal Relations

Universal scalings

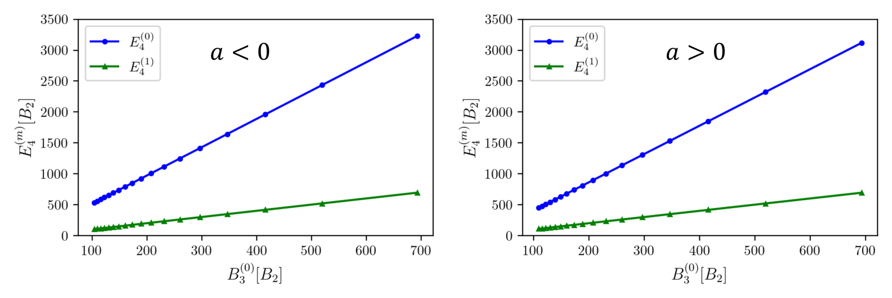


Figure 16: Correlations between trimer and tetramer binding energies; Tjon line* in nuclear physics. a: 2-Boson scattering length.

	$E_4^{(0)}/B_3^{(0)}$	$E_4^{(1)}/B_3^{(0)}$
This work	**4.60(1)**	**1.0022(3)**
Deltuva 2010	4.6108	1.00228

Table 3: Universal scalings in the unitary limit.

*Tjon 1975

Four-body results near unitary limit

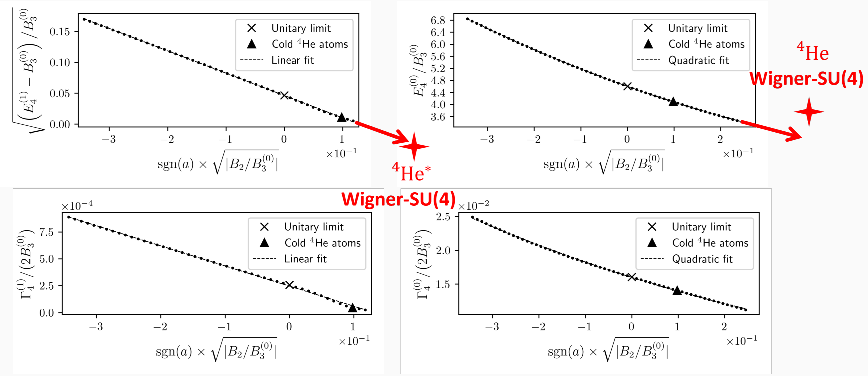


Figure 17: Universal relations between trimer and tetramer binding energies

Four-body results near unitary limit

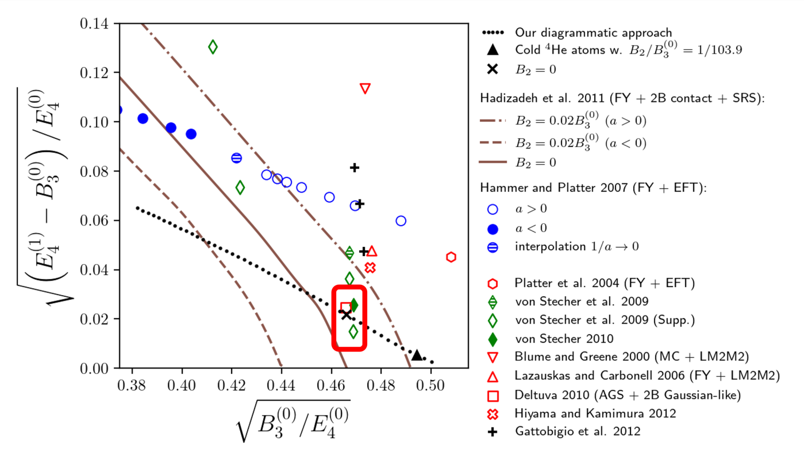


Figure 18: Four-body scaling function*

Hadizadeh et al. 2011

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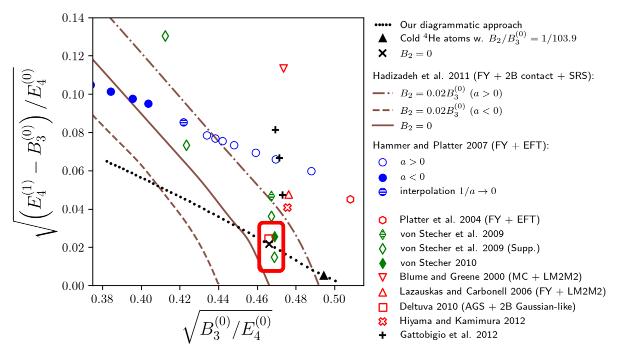


	$E_4^{(0)}/B_3^{(0)}$	$\Gamma_4^{(0)}/(2B_3^{(0)})$	$E_4^{(1)}/B_3^{(0)}$	$\Gamma_4^{(1)}/(2B_3^{(0)})$
This work	**4.60(1)**	**0.0160(1)**	**1.0022(3)**	**$2.57(2)\times 10^{-4}$**
Deltuva 2010	4.6108	0.01484	1.00228	2.38×10^{-4}
von Stecher 2010	4.55	-	1.003	-
von Stecher et al. 2009 (Supp., with V_{3b})	4.55	-	1.001	-

Four-body results near unitary limit

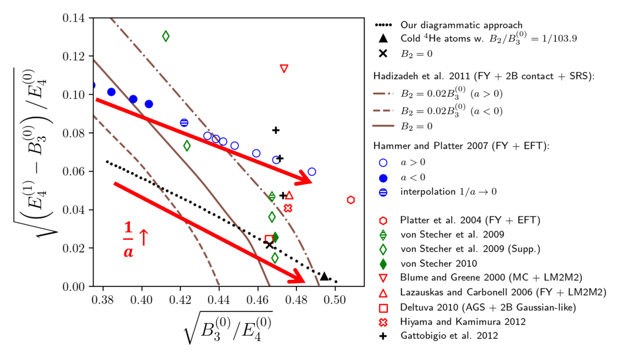


Figure 18: Four-body scaling function*

- Similar flow from unitary limit to cold ^{4}He atoms;
- Origin of difference?

*Hadizadeh et al. 2011

Comparing with FY results

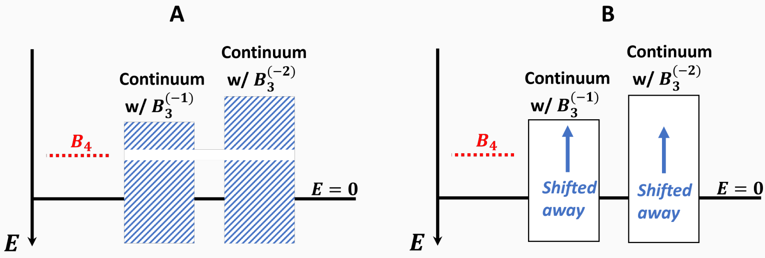
Dealing with trimer poles (Method B)

Method B:* **exclude unwanted deep-trimer spectrum** by shifting them away

$$\hat{V}_3 = |p\rangle V_3 \langle q|$$

$$\rightarrow \hat{\hat{V}}_3 = |p\rangle V_3 \langle q| + |\psi\rangle \eta \langle \psi|$$

$$\Rightarrow \lim_{\eta \to \infty} \hat{\hat{G}}_3 = \hat{G}_3 - \frac{|\psi\rangle R_\psi \langle \psi|}{E + B_3}$$



4B spectrum after subtraction

*See, e.g., Lehman 1982

Comparison with FY results in unitary limit

	$E_4^{(0)}/B_3^{(0)}$	$E_4^{(1)}/B_3^{(0)}$
Lin 2024	4.60(1)	1.0022(3)
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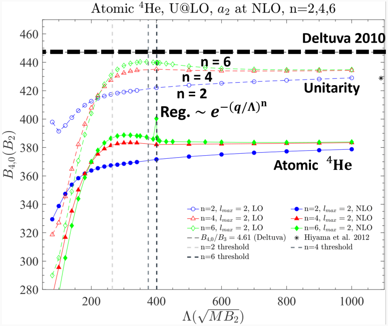

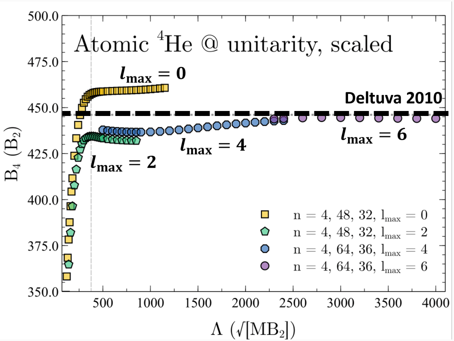

Figure 19: $B_4^{(0)}$ by Faddeev-Yakubovsky equation. Deep trimers shifted by $\eta|\psi\rangle\langle\psi|$. Calculations and figures by Wu, Koenig, and van Kolck.

Conclusion and Outlook

Summary:

- LO and NLO tetramer binding energies of cold ^{4}He atoms (at large cutoffs);
- 4B force needed starting at NLO in contact EFT;
- 4B universal relations;

Outlook:

- Scattering
- Unitary limit $\rightarrow$ finite a $\rightarrow$ 4-nucleon system (adding Wigner-SU(4)-breaking NN interaction)
- Halo/Cluster systems (suggestion?)

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