Ambipolar diffusion operator in neutron star cores

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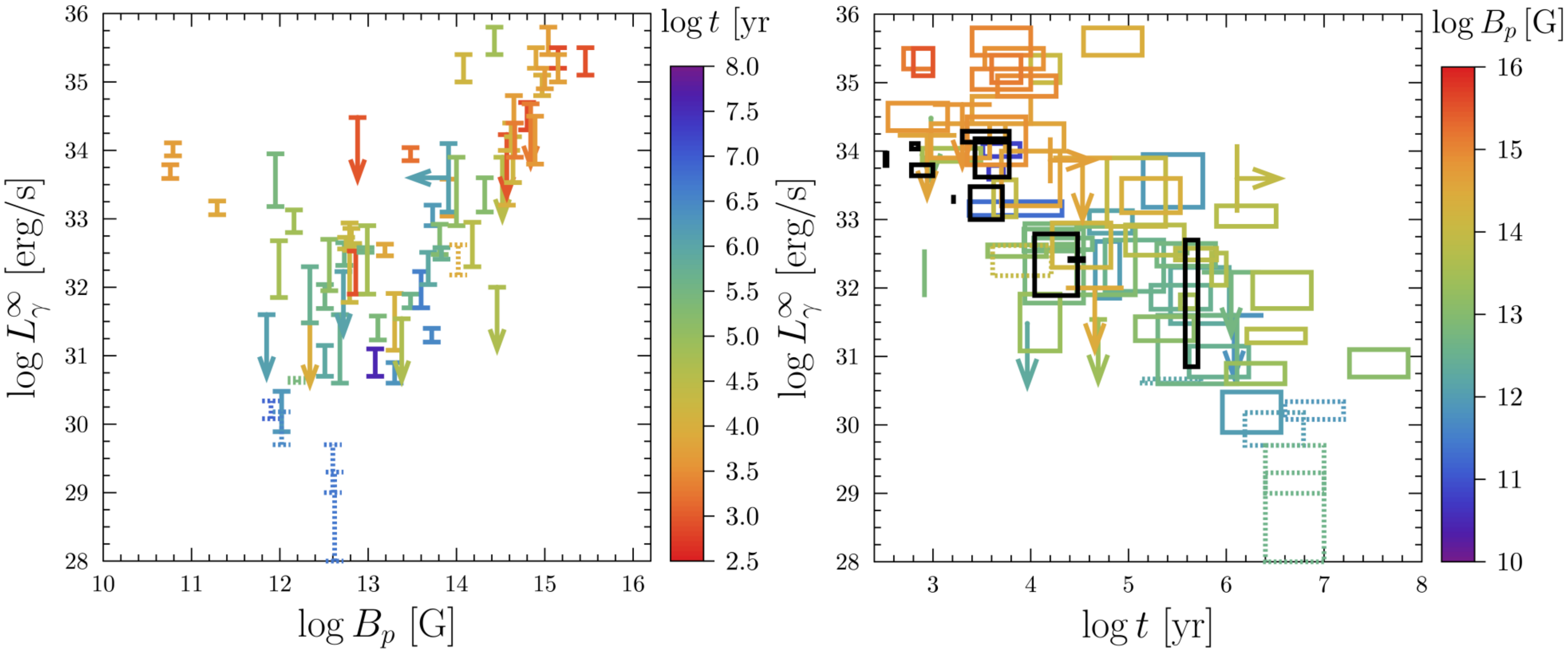
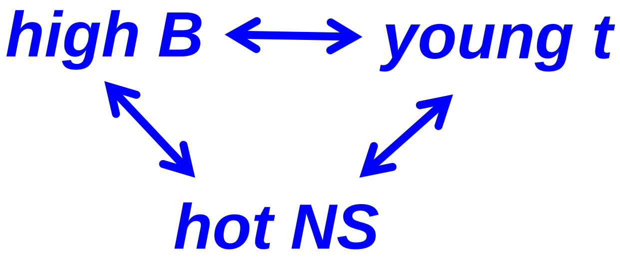
NS Magneto-Thermal States



Data collection:

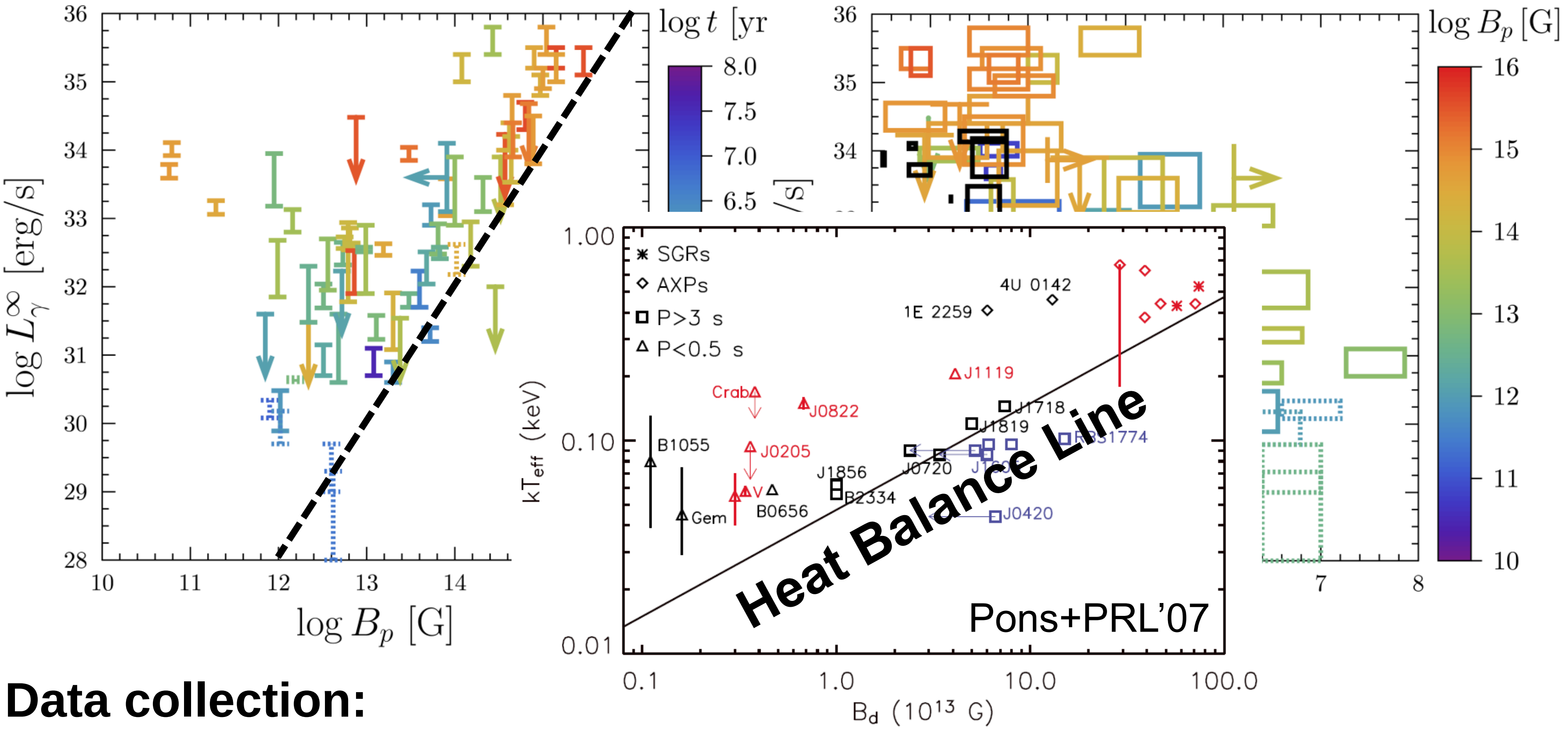
- ➢ *Potekhin+'20, Vigano+'13, Coti Zelati+'18,…*
- ➢ *69 NSs with "known" t, **thermal** L_γ^∞, B_p*
  - ○ *10 NSs with t and L_γ^∞ only*



high B ⟷ ***young t***

hot NS

NS Magneto-Thermal States



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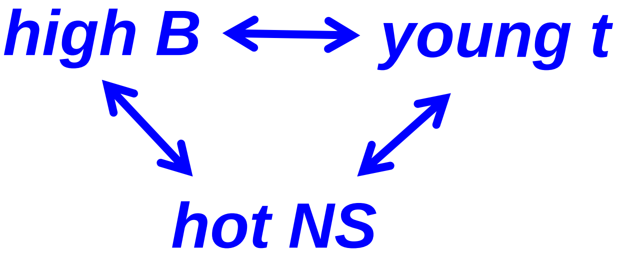
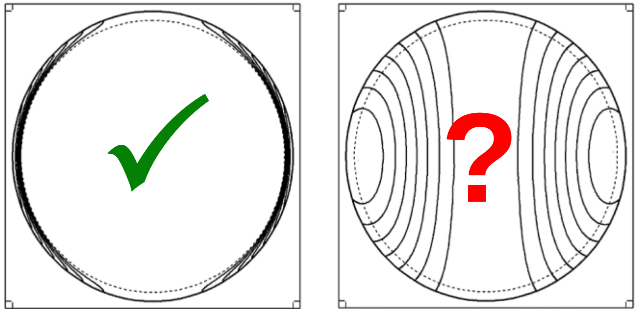


Dissipation of Magnetic Field

- **crust**

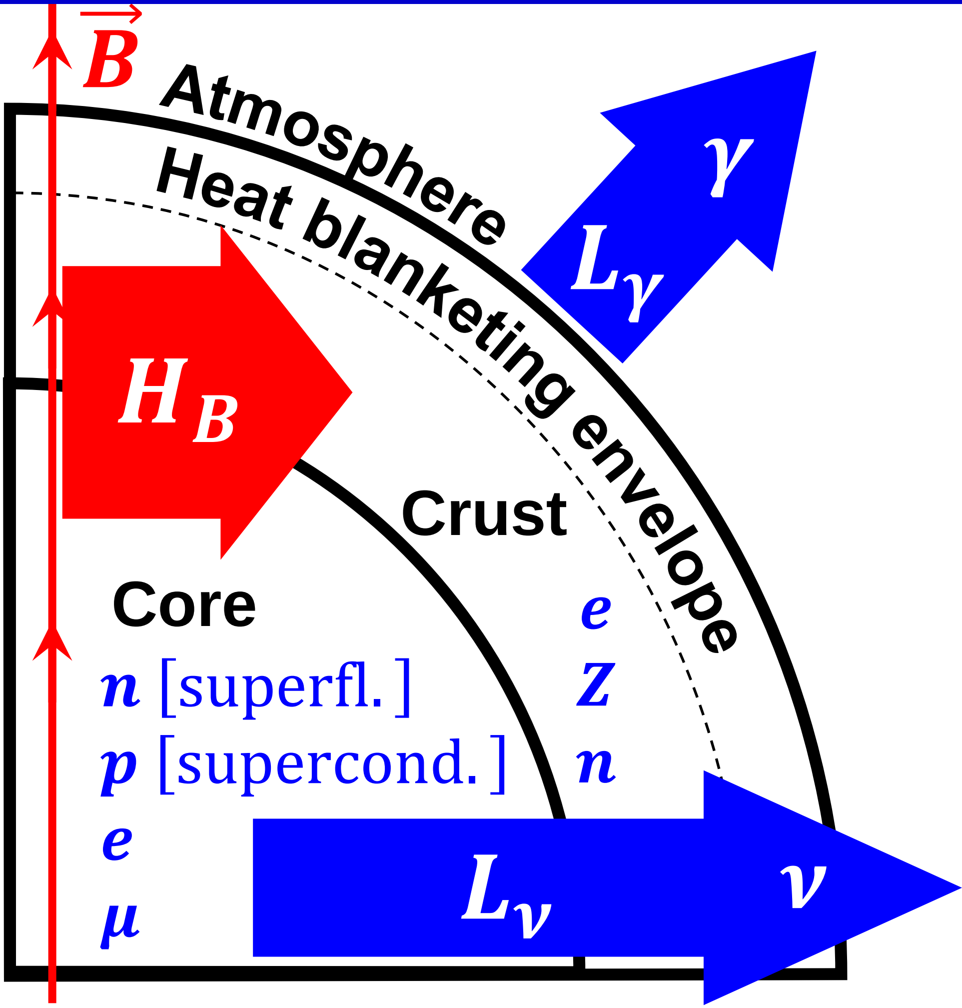
Pons+’07, Vigano+’13, DeGrandis+’21, Igoshev&Hollerbach’21,23, Dehman+’21,22,23

- ➢ **eZ scattering *(Ohm)* + *Hall***



- **core**

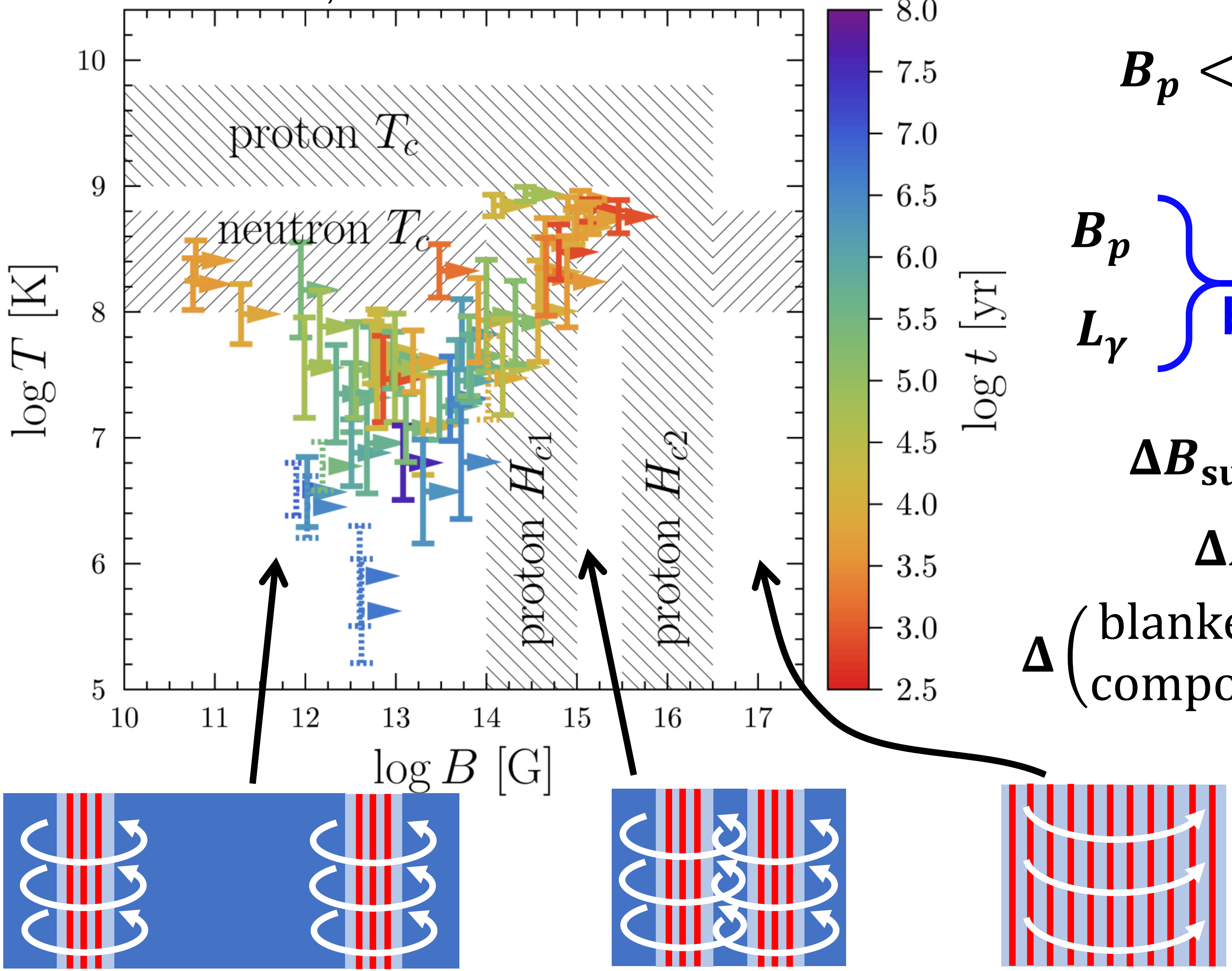
Goldreich&Reisenegger’92, Castillo+’20, Passamonti+’17, Moraga+’24, Elfritz+’16, Bransgrove+’18, Igoshev&Hollerbach’23, Dehman+’21,22,23, Gusakov,Kantor&DO+’17, DO&Gusakov’18, Gusakov,Kantor&DO’20,…, ***this talk***



- ➢ **$npe\mu$ scattering *(diffusion)***
- ➢ ***nonequilibrium Urca*-processes**
- ➢ **p-pairing ⇒ *scattering off flux tubes***

B and *T* inside NS cores

DO, Gusakov & Potekhin 2023

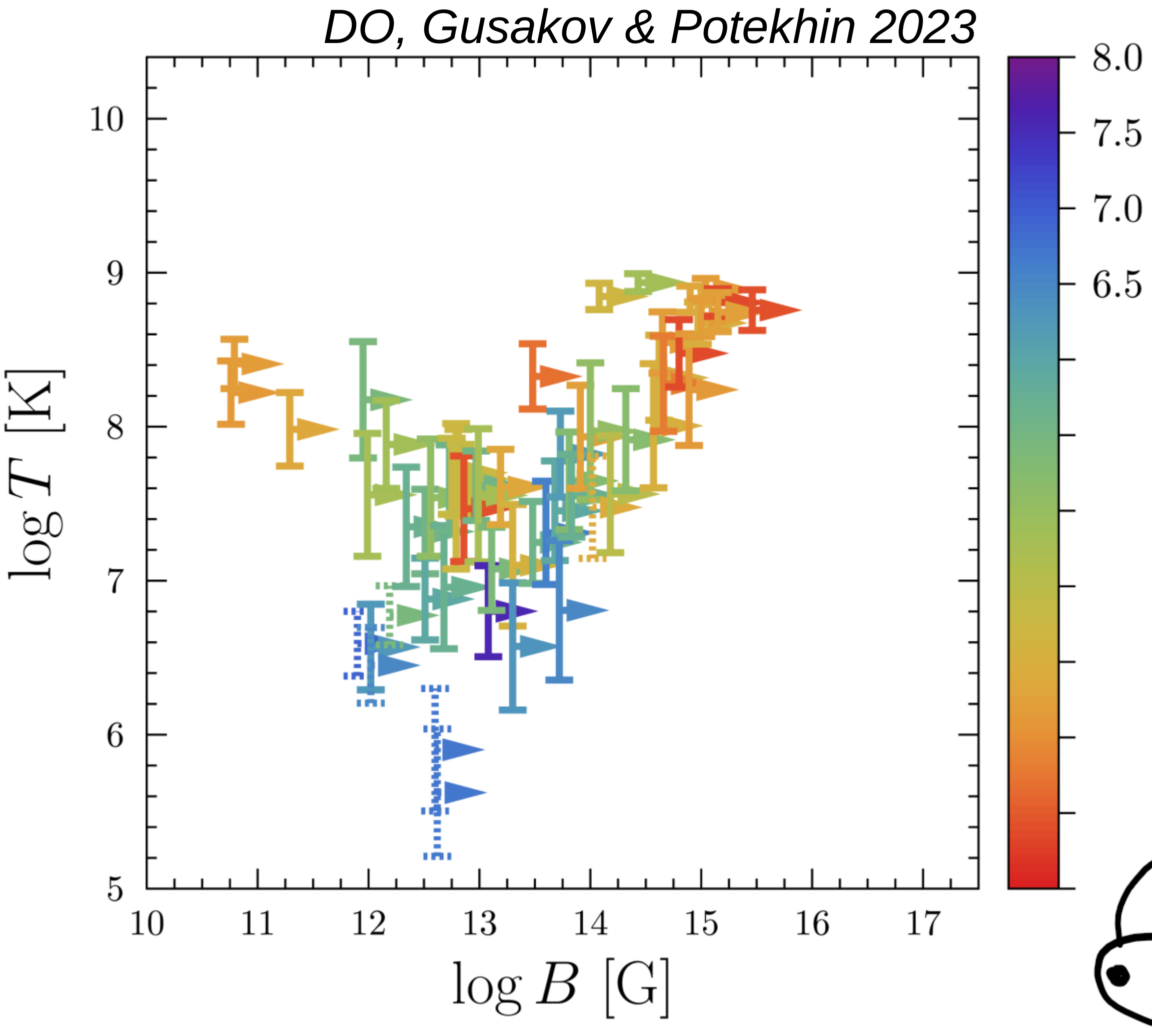


$B_p < B$ **inside**

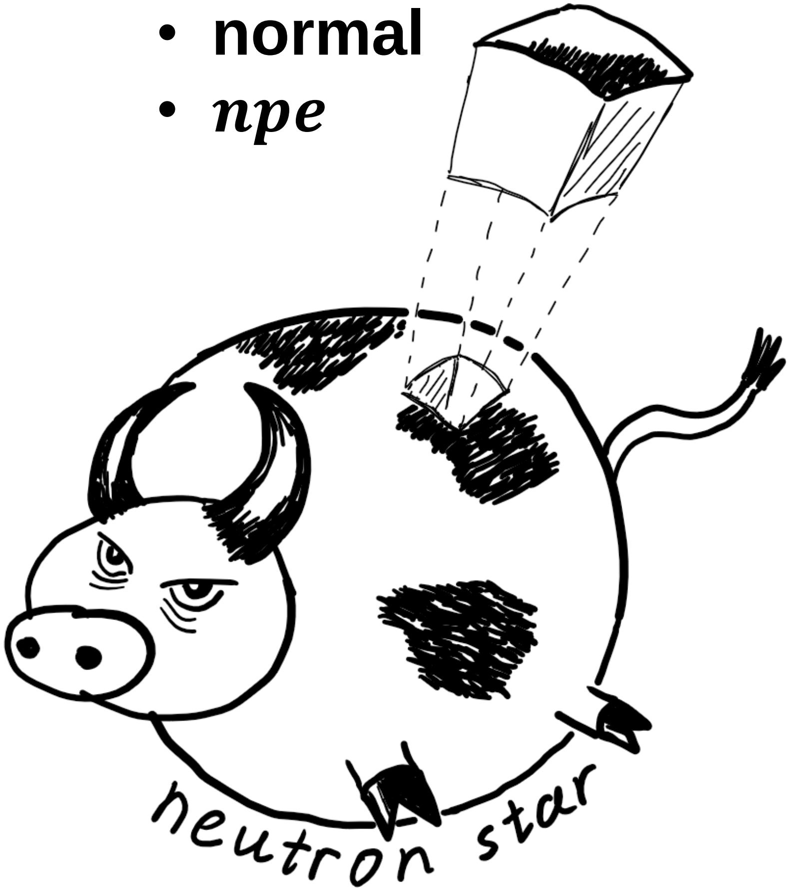
$$\left.\begin{matrix}B_p\\ L_\gamma\end{matrix}\right\}\xrightarrow[\text{blanket}]{\text{heat}} T$$

$$\left.\begin{matrix}\Delta B_{\mathrm{surf}}\\ \Delta L_\gamma\\ \Delta\begin{pmatrix}\text{blanket}\\ \text{compos.}\end{pmatrix}\end{matrix}\right\}\rightarrow \Delta T$$

A Simplistic Approach

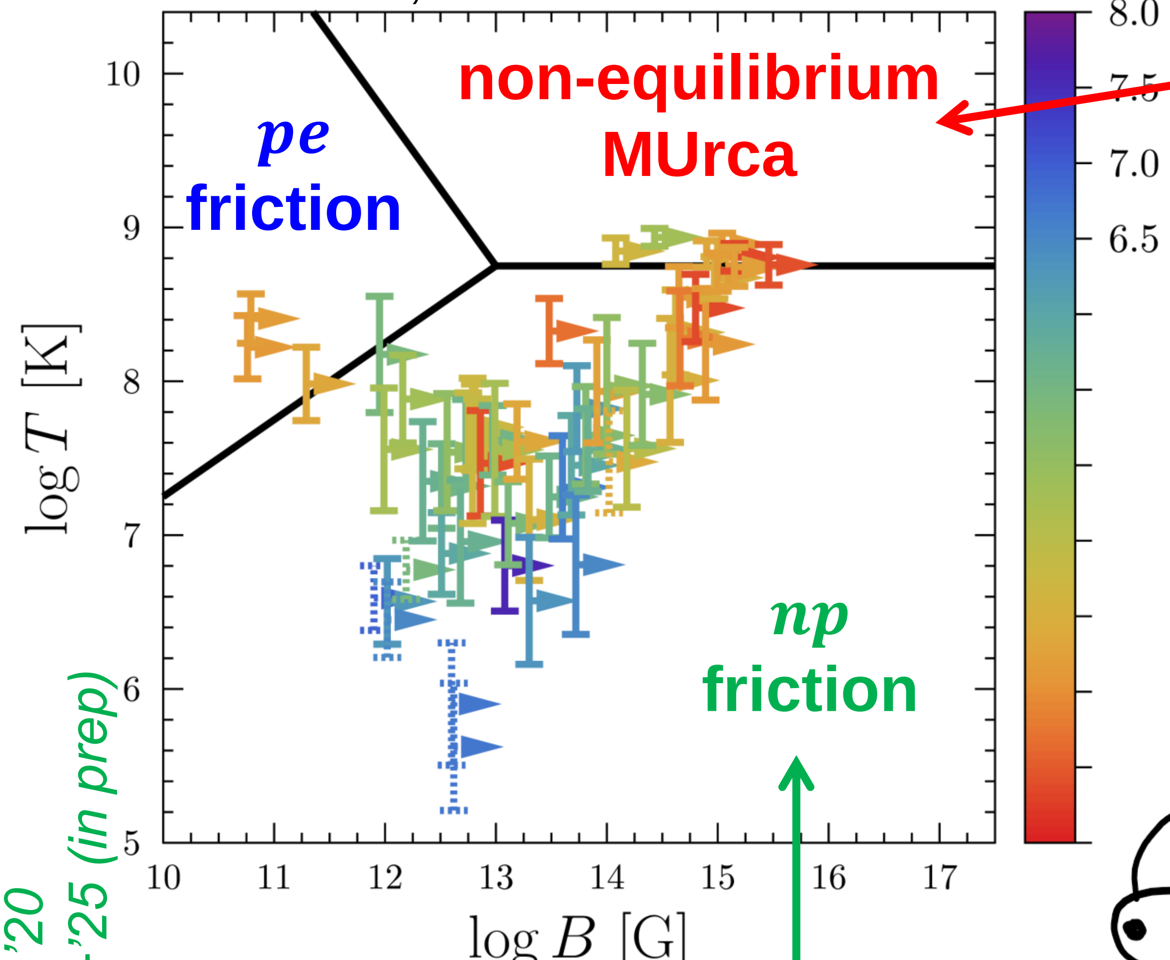


- **Newtonian MHD**
- **normal**
- ***npe***



Dominating Mechanisms

DO, Gusakov & Potekhin 2023

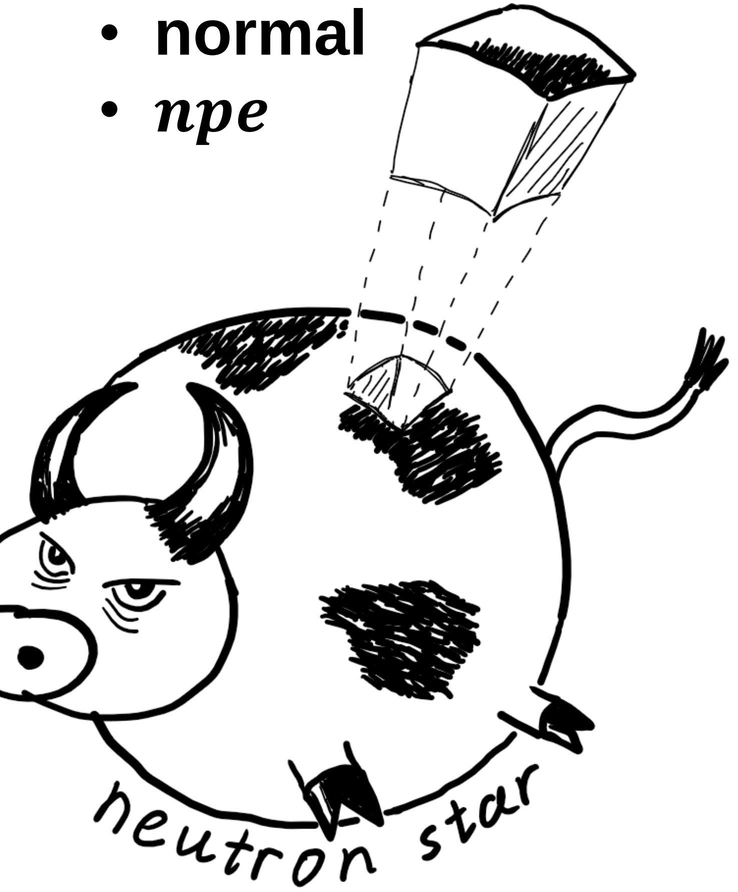


strong-coupling regime

$v_n \approx v_p \approx v_e \approx v$

Moraga+'24

- **Newtonian MHD**
- **normal**
- ***npe***



Castillo+'20
Moraga+'25 (in prep)

weak-coupling regime

$v_n \neq v_p \approx v_e \approx v_c$

ambipolar diffusion

Outline

- **Pons & Vigano (2019, Liv. Rev. Comp. Astrophys):** **a simplistic model for ambipolar diffusion**

$$\vec{f}_L = \frac{1}{4\pi} \mathrm{curl}\, \vec{B} \times \vec{B} \qquad \vec{v}_c = \alpha \vec{f}_L \qquad \partial_t \vec{B} = \mathrm{curl}\left(\vec{v}_c \times \vec{B}\right)$$

velocity of charged species

- **Neutron star cores, normal npe + axisymmetry:**

$$\vec{f}_L = \frac{1}{4\pi} \mathrm{curl}\, \vec{B} \times \vec{B} \qquad \vec{v}_c^{(p)} = \hat{A}^{(p)} \vec{f}_L \qquad \partial_t \vec{B} \text{ is determined by } \vec{v}_c^{(p)}$$

poloidal projection

$\hat{A}^{(p)}$ – ambipolar diffusion operator

- linear & independent of $\vec{B}$
- explicit analytic form
- completely drives $\vec{B}$ evolution
- $\ker \hat{A}^{(p)} =$ equilibrium $\vec{B}$'s
- self-adjoint

Basic Equations

$$n_b \nabla \delta\mu_n + n_c \nabla \Delta\mu = f_L = \frac{1}{4\pi} \operatorname{curl} \vec{B} \times \vec{B} \qquad \sum_{n,p,e} \text{Euler} = \text{force balance}$$

$\delta\mu_p + \delta\mu_e - \delta\mu_n$

$$n_n n_c \gamma (\vec{v}_c - \vec{v}_n) = n_n \nabla \delta\mu_n \qquad \text{Euler } n$$

np friction

$$-e n_c \left(E + \frac{\vec{v}_c}{c} \times \vec{B} \right) = n_c \nabla \delta\mu_e \qquad \text{Euler } e$$

$$\operatorname{div} n_n \vec{v}_n = \operatorname{div} n_c \vec{v}_c = 0 \qquad \text{Continuity}$$

$$\partial_t \vec{B} = -c \operatorname{curl} E = \operatorname{curl}\left(\vec{v}_c \times \vec{B} \right) \qquad \text{Faraday}$$

Linearization

$\vec{v}_{n,c}$, $\delta\mu_{n,p,e}$ as small perturbations

$n_{n,c}$ as spherical TOV background

Quasistationarity

$\partial_t = 0$ except $\partial_t \vec{B}$

Weak Coupling Regime

no reactions

Ambipolar Diffusion

$\vec{v}_e = \vec{v}_p = \vec{v}_c \neq \vec{v}_n$

$\vec{j} = 0$ but $\operatorname{curl} \vec{B} \times \vec{B} \neq 0$

Cowling approximation

Basic Equations

$$n_b\nabla\delta\mu_n + n_c\nabla\Delta\mu = f_L = \frac{1}{4\pi}\,\mathrm{curl}\,\vec{B}\times\vec{B}$$

$\sum_{n,p,e}$ Euler = force balance

$\delta\mu_p + \delta\mu_e - \delta\mu_n$

$$n_n n_c \gamma(\vec{v}_c - \vec{v}_n) = n_n\nabla\delta\mu_n$$

Euler n

np friction

$$-en_c\left(E + \frac{\vec{v}_c}{c}\times\vec{B}\right) = n_c\nabla\delta\mu_e$$

Euler e

$$\mathrm{div}\,n_n\vec{v}_n = \mathrm{div}\,n_c\vec{v}_c = 0$$

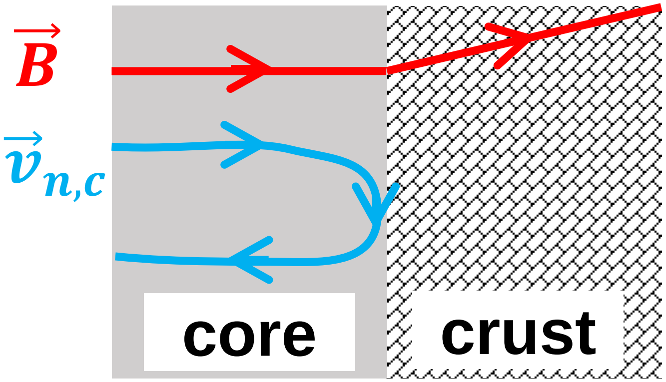
Continuity

$$\partial_t\vec{B} = -c\,\mathrm{curl}\,E = \mathrm{curl}\big(\vec{v}_c\times\vec{B}\big)$$

Faraday

Boundary Conditions

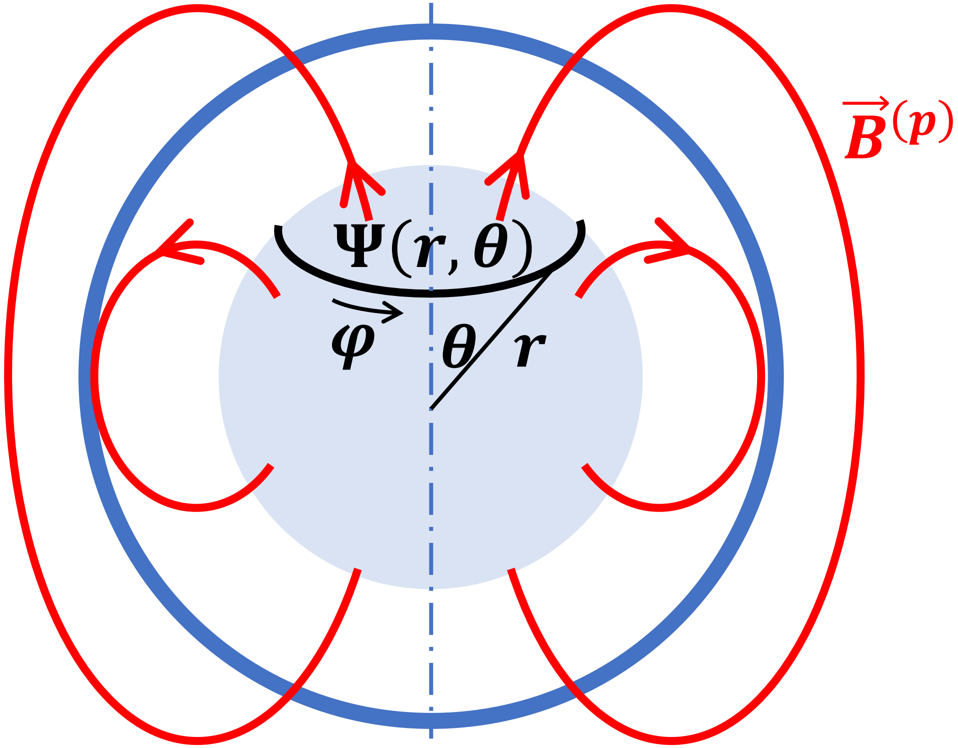
- $\vec{B}$ continuously matches the crust
- $\vec{v}_n$ and $\vec{v}_c$ do not come into the crust
- no $\vec{j}$ matching!

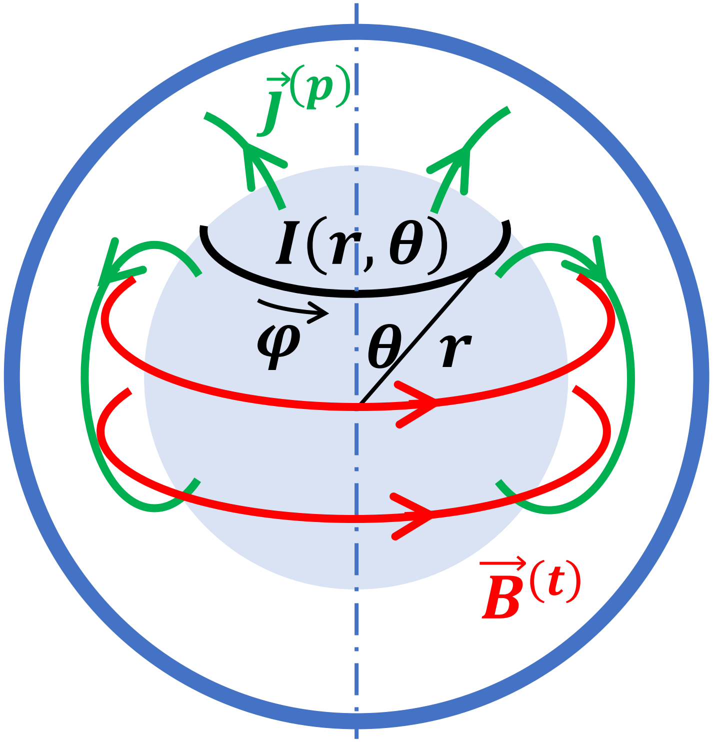


Axial Symmetry

$$\vec{B} = \nabla\Psi(r,\theta) \times \nabla\varphi + I(r,\theta)\nabla\varphi$$

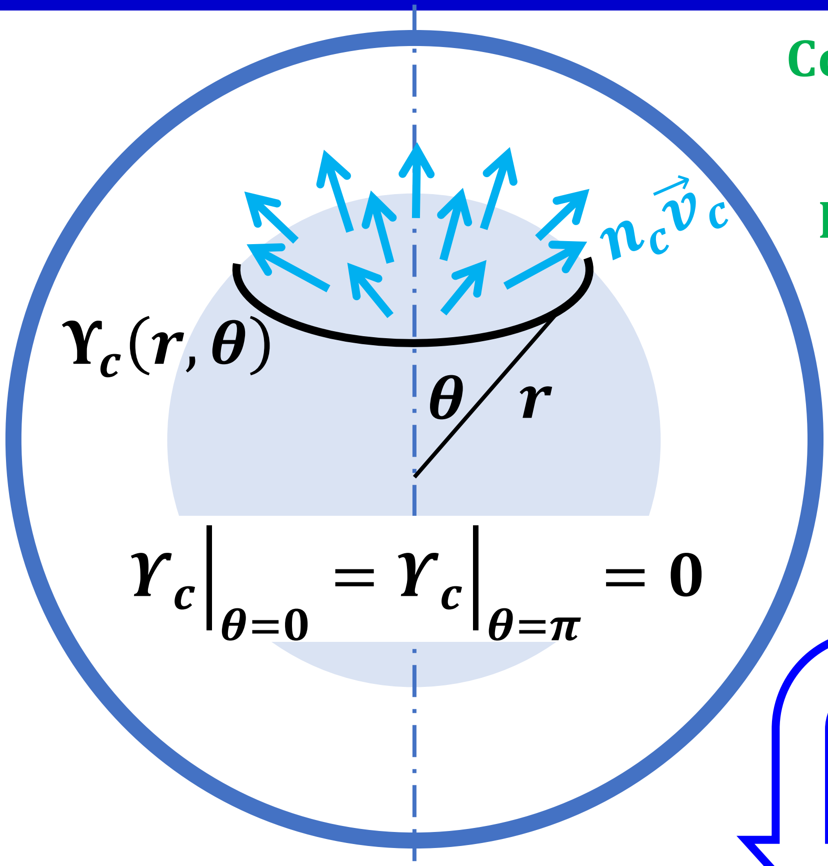
Poloidal flux function $\leftrightarrow \vec{B}^{(p)}$ **Poloidal current function** $\leftrightarrow \vec{B}^{(t)}$





$$n_b \nabla\delta\mu_n(r,\theta) + n_c\nabla\Delta\mu(r,\theta) = \vec{f}_L \Rightarrow f_{L\varphi} = 0 \Rightarrow I = I(\Psi, t)$$

Magic of Axisymmetry - I



$$\Upsilon_c\Big|_{\theta=0} = \Upsilon_c\Big|_{\theta=\pi} = 0$$

Continuity – c

$$n_c\vec{v}_c^{(p)} = \nabla\Upsilon_c(r,\theta)\times\nabla\varphi$$

Euler – n

$$n_n\vec{v}_n = n_n\vec{v}_c - \frac{n_n}{n_c\gamma}\nabla\delta\mu_n$$

$$\mathrm{div}\, n_n\vec{v}_n = 0$$

Continuity – n

$n_{n,c} = \mathrm{const}(\theta)$

$$\frac{1}{\sin\theta}\partial_\theta\Upsilon_c = r^2\mathcal{R}\,\mathrm{div}\left(\frac{1-Y}{\gamma Y}\nabla\delta\mu_n\right)$$

$$Y(r) = \frac{n_c}{n_n+n_c} \qquad \mathcal{R}(r) = \left(\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{Y}\right)^{-1}$$

$$\sin\theta\,\partial_\theta\frac{1}{\sin\theta}\partial_\theta\Upsilon_c = \mathcal{R}\left[\partial_r\left(r^2\frac{1-Y}{\gamma Y}\partial_r(\sin\theta\,\partial_\theta\delta\mu_n)\right) + \frac{1-Y}{\gamma Y}\sin\theta\,\partial_\theta\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\delta\mu_n)\right]$$

Magic of Axisymmetry - II

$$\mathbf{curl}\left(\frac{\textbf{force balance}}{n_c}\right): \qquad \nabla\left(\frac{n_b}{n_c}\right)\times\nabla\delta\mu_n = \mathbf{curl}\left(\frac{\vec{f}_\mathrm{L}}{n_c}\right)$$

$$\Downarrow$$

$$\sin\theta\,\partial_\theta\delta\mu_n = \mathcal{R}r\sin\theta\left[\mathbf{curl}\left(\frac{\vec{f}_\mathrm{L}}{n_c}\right)\right]_\varphi$$

$$\begin{tikzcd}
\Upsilon_c\Big|_{\theta=0} = \Upsilon_c\Big|_{\theta=\pi} = 0 \arrow[dr] & & \\
& \exists(\Delta_\theta^*)^{-1} \arrow[r] & \exists\hat{U}_c\colon \Upsilon_c = \hat{U}_c\left(\mathcal{R}r\sin\theta\left[\mathbf{curl}\left(\frac{\vec{f}_\mathrm{L}}{n_c}\right)\right]_\varphi\right) \\
\Delta_\theta^* \arrow[ur] & &
\end{tikzcd}$$

$$\overbrace{\sin\theta\,\partial_\theta\frac{1}{\sin\theta}\partial_\theta}^{\Delta_\theta^*}\Upsilon_c = \mathcal{R}\left[\partial_r\left(r^2\frac{1-Y}{\gamma Y}\partial_r(\sin\theta\,\partial_\theta\delta\mu_n)\right) + \frac{1-Y}{\gamma Y}\sin\theta\,\partial_\theta\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta\delta\mu_n)\right]$$

Explicit solution for $v_c^{(p)}$

$$\vec{v}_c^{(p)} = \hat{A}^{(p)}\vec{f}_L = \hat{V}_c\hat{U}_c\hat{F}\vec{f}_L \quad \begin{cases} \hat{V}_c = -\dfrac{\nabla\varphi}{n_c} \times \nabla \\ \hat{U}_c = \mathcal{R}\left[\partial_r\left(r^2\dfrac{1-Y}{\gamma Y}\partial_r \Delta_\theta^{*\,-1}\right) + \dfrac{1-Y}{\gamma Y}\right] \\ \hat{F} = \mathcal{R} r \sin\theta \left[\operatorname{curl}\left(\dfrac{\cdot}{n_c}\right)\right]_\varphi \end{cases}$$

$$Y(r) = \frac{n_c}{n_n + n_c} \qquad \mathcal{R}(r) = \left(\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{Y}\right)^{-1}$$

- **linear**
- r**-local;** $\partial_r^{(5)} B$, $\partial_r^{(6)}\Psi$
- θ**-nonlocal**
- $\gamma \propto T^2 \Rightarrow \hat{A}^{(p)} \propto T^{-2}$

- $Y \to \text{const} \Rightarrow \mathcal{R} \to \infty \Rightarrow$ **Passamonti+'17 @ low *T***

$$\operatorname{div}\left(\frac{1-Y}{\gamma Y}\nabla\delta\mu_n\right) = 0$$

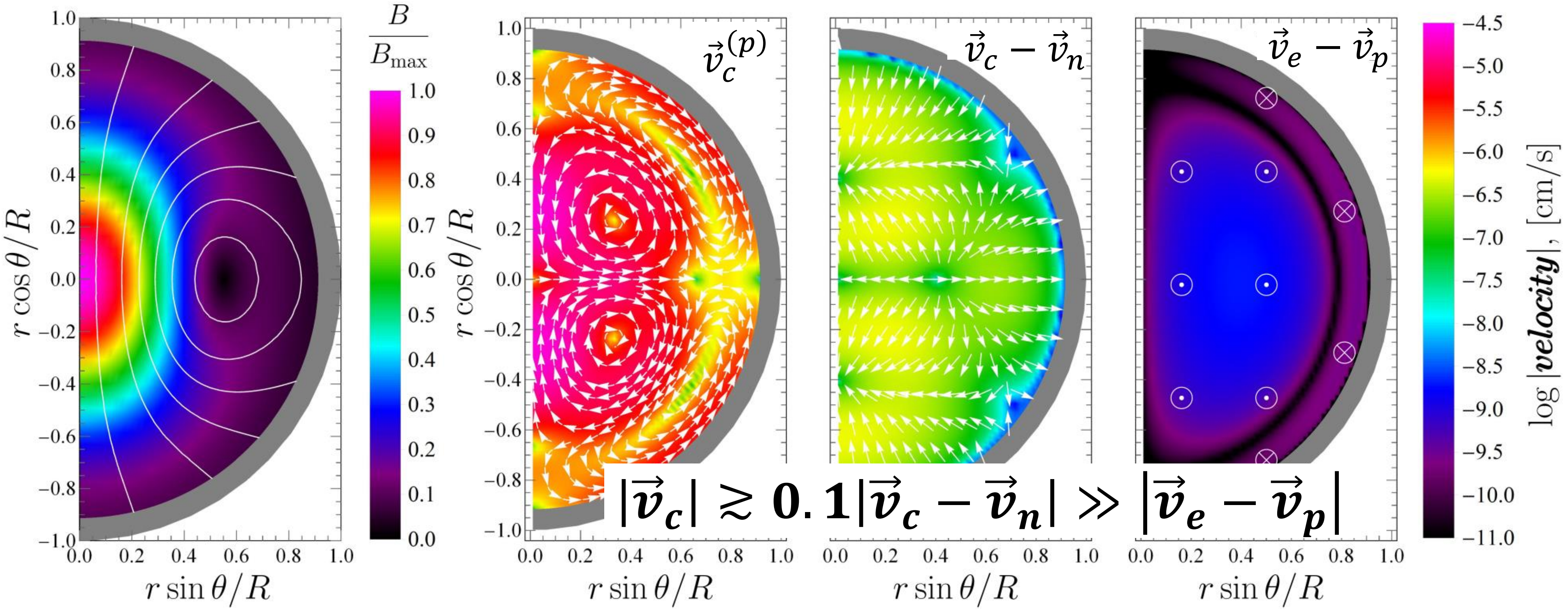
How it (should) work

$$\vec{v}_c^{(p)} = \hat{A}^{(p)}\vec{f}_L = \hat{V}_c\hat{U}_c\hat{F}\vec{f}_L$$

$$\vec{v}_e - \vec{v}_p = -\frac{\mathbf{curl}\,\vec{B}}{4\pi e n_c}$$

$$\exists \hat{V}_n, \hat{U}_n:\ \ \vec{v}_n^{(p)} = \hat{V}_n\hat{U}_n\hat{F}\vec{f}_L$$

$$B_{\max} = 5\ \times 10^{15}\mathrm{G}, \qquad T = 2\times 10^{8}\mathrm{K}$$



$$|\vec{v}_c| \gtrsim 0.1|\vec{v}_c - \vec{v}_n| \gg |\vec{v}_e - \vec{v}_p|$$

Theorem I: $\widehat{A}^{(p)}$ determines $\vec{B}$ evolution

$$\begin{cases} \partial_t \vec{B}^{(p)} = \mathbf{curl}\left(\vec{v}_c^{(p)} \times \vec{B}^{(p)}\right) \\ \partial_t \vec{B}^{(t)} = \mathbf{curl}\left(\vec{v}_c^{(p)} \times \vec{B}^{(t)} + \vec{v}_c^{(t)} \times \vec{B}^{(p)}\right) \end{cases}$$

$$\vec{v}_c^{(p)} = \widehat{A}^{(p)} \vec{f}_L$$

$$\vec{v}_c^{(t)} = ?$$

$$\vec{B}^{(p)} = \nabla\Psi(r,\theta) \times \nabla\varphi$$

$$\vec{B}^{(t)} = I(\Psi, t)\nabla\varphi$$

$$\int dV (\nabla\varphi)^2 I(\Psi) g(\Psi) = \mathbf{const}(t)$$

Magnetic helicity between Ψ_1 and Ψ_2 = const

$$\begin{cases} \partial_t \Psi = -\nabla\Psi \cdot \widehat{A}^{(p)} \vec{f}_L \\ I(\Psi, t) = I(\Psi, 0) \dfrac{\left. \int_\Psi d\ell\, (\nabla\varphi)^2 / B^{(p)} \right|_{t=0}}{\left. \int_\Psi d\ell\, (\nabla\varphi)^2 / B^{(p)} \right|_{t}} \end{cases}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Theorem II:
$\ker \hat{A}^{(p)}$ = Grad-Shafranov fields

- **Grad-Shafranov equilibrium & Grad-Shafranov equation**

$$\vec{f}_L = n_c \nabla \Delta\mu \;\Leftrightarrow\; \Delta^*\Psi + I(\Psi)I'(\Psi) + 4\pi n_c r^2 \sin^2\theta\,\Delta\mu(\Psi) = 0$$

- $\Longrightarrow$

$$\operatorname{curl}\frac{\vec{f}_L}{n_c} = 0 \;\Rightarrow\; \vec{v}_c^{(p)} = \hat{V}_c \hat{U}_c\left(\mathcal{R} r \sin\theta \left[\operatorname{curl}\left(\frac{\vec{f}_L}{n_c}\right)\right]_\varphi\right) = 0$$

$$\Downarrow$$

$$\boxed{\{\mathrm{GS}\} \subset \ker \hat{A}^{(p)}}$$

- $\Longleftarrow$

$$]\; \vec{B}(r) \in \ker \hat{A}^{(p)} \;\Rightarrow\; \vec{v}_c^{(p)} = 0 \;\Rightarrow\; \partial_t \vec{B} = 0 \;\Rightarrow\; \boxed{\ker \hat{A}^{(p)} \subset \{\mathrm{GS}\}}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Key Quantity for Evolution

$\xcancel{B}$ $\xcancel{f_L}$ ✓ $\Phi = \mathcal{R} r \sin\theta \left[\mathrm{curl} \left(\dfrac{\vec{f}_L}{n_c} \right) \right]_{\varphi}$

$$\begin{tikzcd}
 & \Phi = \mathcal{R} r \sin\theta \left[\mathrm{curl} \left(\dfrac{\vec{f}_L}{n_c} \right) \right]_{\varphi} \arrow[dl] \arrow[dr] & \\
\Phi = 0 & & \Phi \neq 0
\end{tikzcd}$$

no evolution

evolution rate ∝ Φ

$$\vec{B} \longrightarrow \Phi \longrightarrow \vec{v}_c^{(p)} = \widehat{V}_c \widehat{U}_c \Phi \longrightarrow \partial_t \vec{B} = \mathrm{curl}\left(\vec{v}_c \times \vec{B}\right)$$

~~curvature of $\vec{B}$ lines~~

curvature of $\vec{f}_L$ lines ✓

Dissipation of Magnetic Field

$$\dot{W}_B = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}V\,\frac{B^2}{8\pi} = -H_R - H_{np} - H_{pe}$$

$$H_R = \int \mathrm{d}V\,\lambda\Delta\mu^2 \propto B^4T^6$$

nonequilibrium [modified] Urca reactions

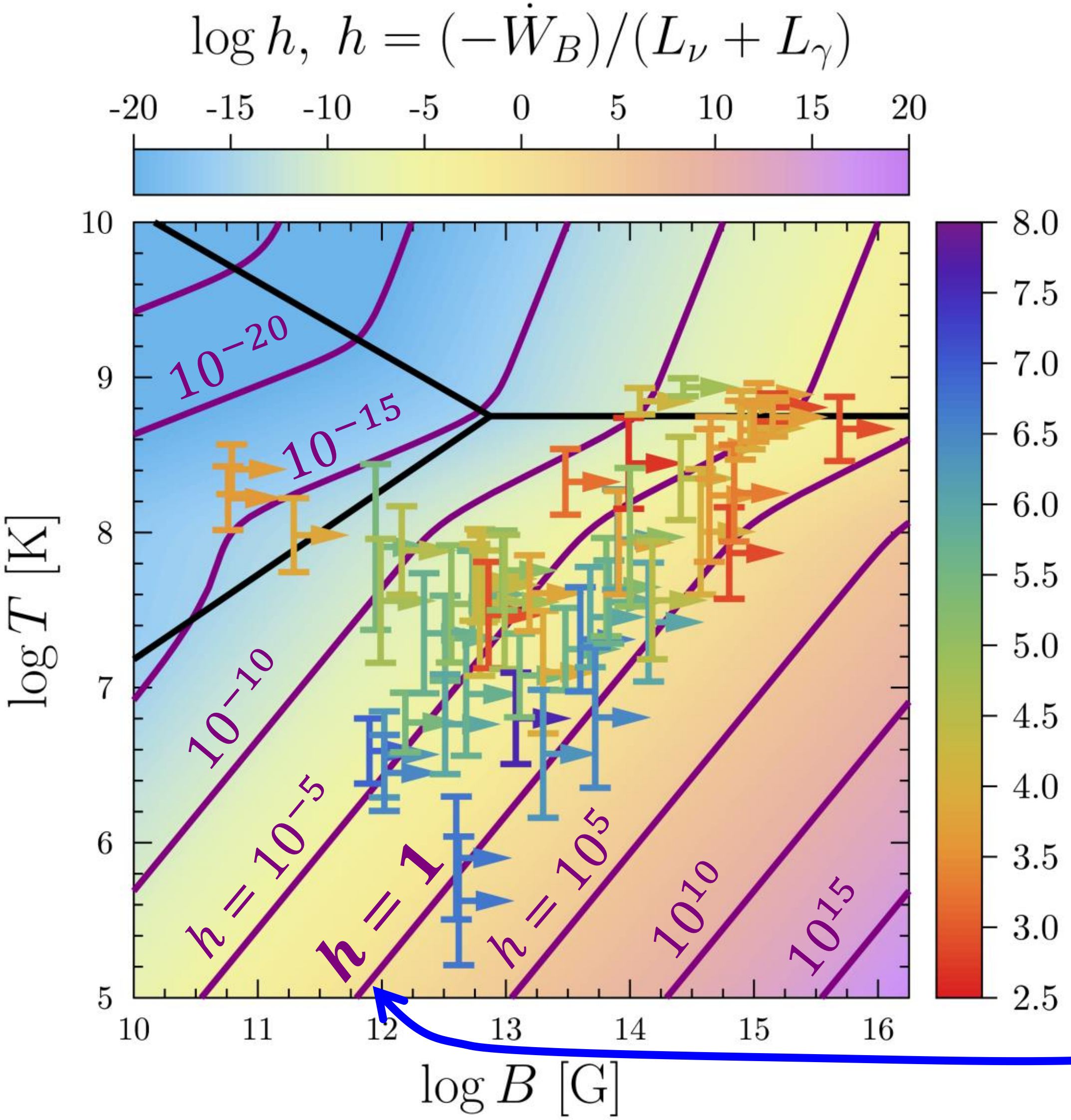
$$H_{np} = \int \mathrm{d}V\; n_n n_c \gamma\left(\vec{v}_n - \vec{v}_p\right)^2 = \int \mathrm{d}V\vec{f}_L \cdot \widehat{A}^{(p)}\vec{f}_L \propto \frac{B^4}{T^2}$$

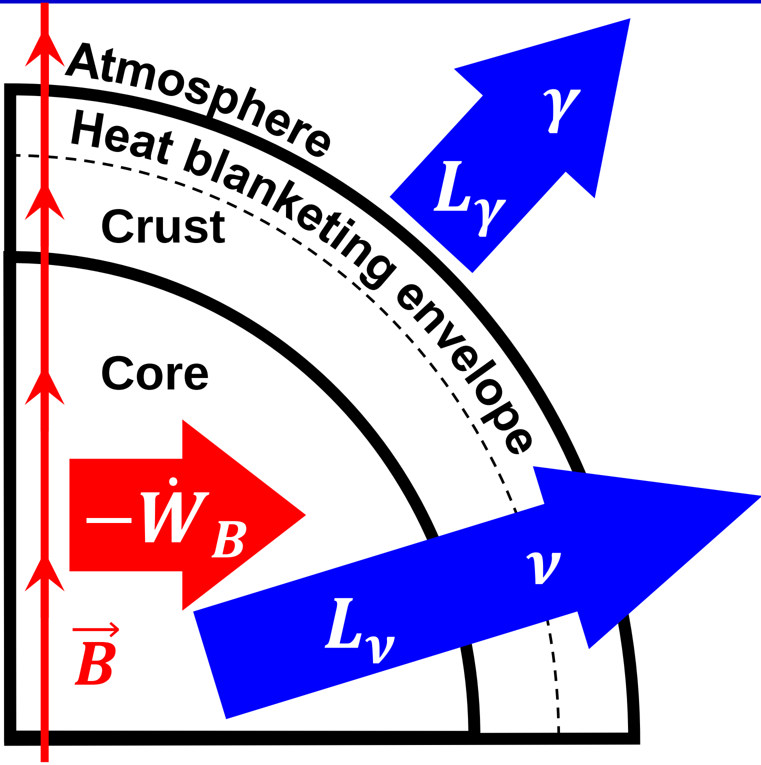
strong-force friction

$$H_{pe} = \int \mathrm{d}V\, n_c^2 \gamma_{pe}\left(\vec{v}_p - \vec{v}_e\right)^2 = \int \mathrm{d}V\,\gamma_{pe}\left(\frac{c\,\mathrm{curl}\,\vec{B}}{4\pi e}\right)^2 \propto B^2T^{5/3\ldots2}$$

electromagnetic friction

Heating vs Cooling: Naive Estimate





$$\dot{W}_B + L_\nu + L_\gamma = 0$$

heat balance line

Here: **from $\partial_t \vec{B}$ in the core**

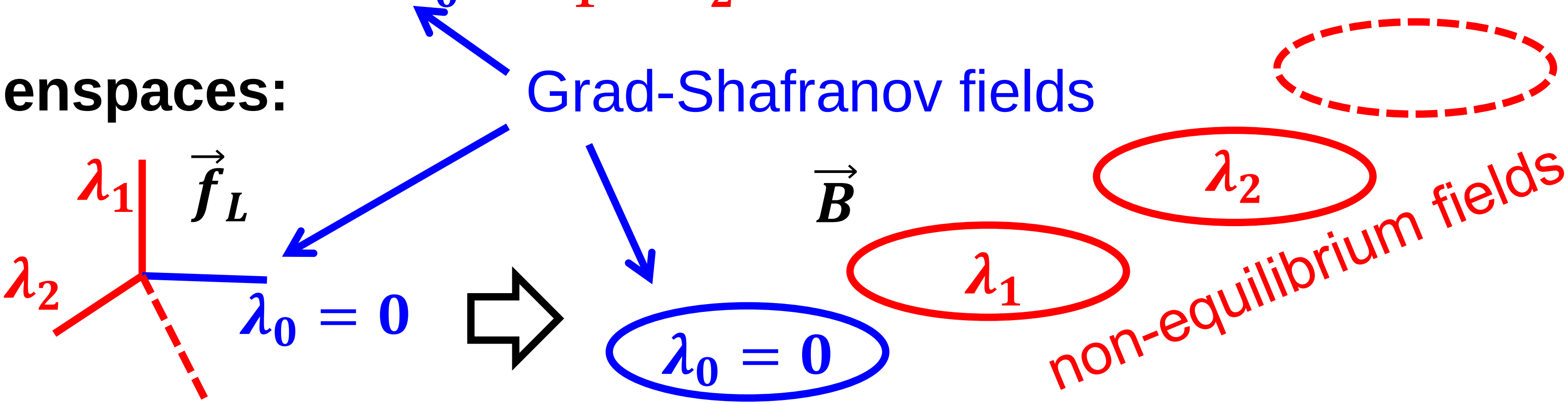
Pons+'07: from $\partial_t \vec{B}$ in the crust

Theorem III: $\widehat{A}^{(p)}$ is self-adjoint

$$H_{np} = \int \mathrm{d}V \vec{f}_L \cdot \widehat{A}^{(p)} \vec{f}_L = \left\langle \vec{f}_L \middle| \widehat{A}^{(p)} \vec{f}_L \right\rangle \geq 0$$

$$\forall \vec{f}_{L1}, \vec{f}_{L2}: \quad \left\langle \vec{f}_{L1} \middle| \widehat{A}^{(p)} \vec{f}_{L2} \right\rangle = \left\langle \vec{f}_{L2} \middle| \widehat{A}^{(p)} \vec{f}_{L1} \right\rangle \qquad \mathcal{Q}.\mathcal{E}.\mathcal{D}.$$

- **eigenvalues:** $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$
- **eigenspaces:**



More Realistic NSs

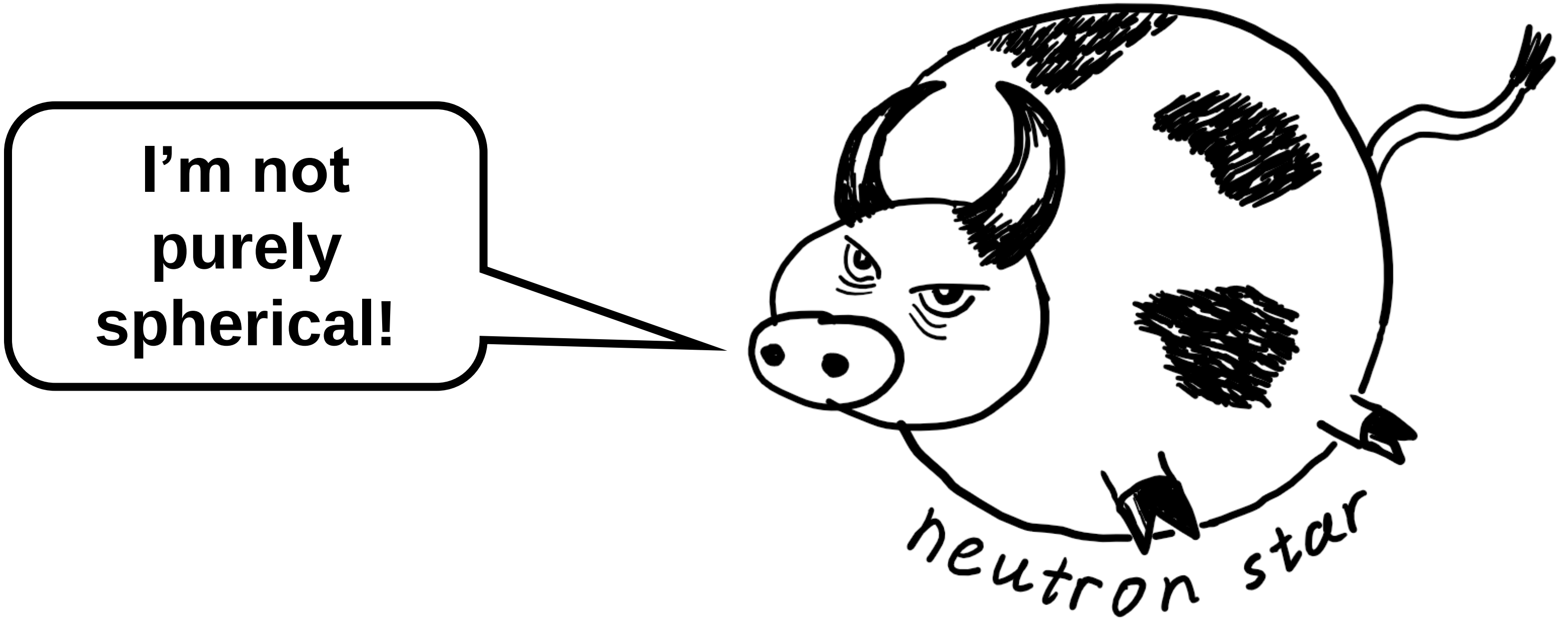
GR MHD

only quantitative changes

npeμ+

$\widehat{A}^{(p)}$ **exists, but** $= \mathrm{var}\left(\vec{B}\right)$

pairing



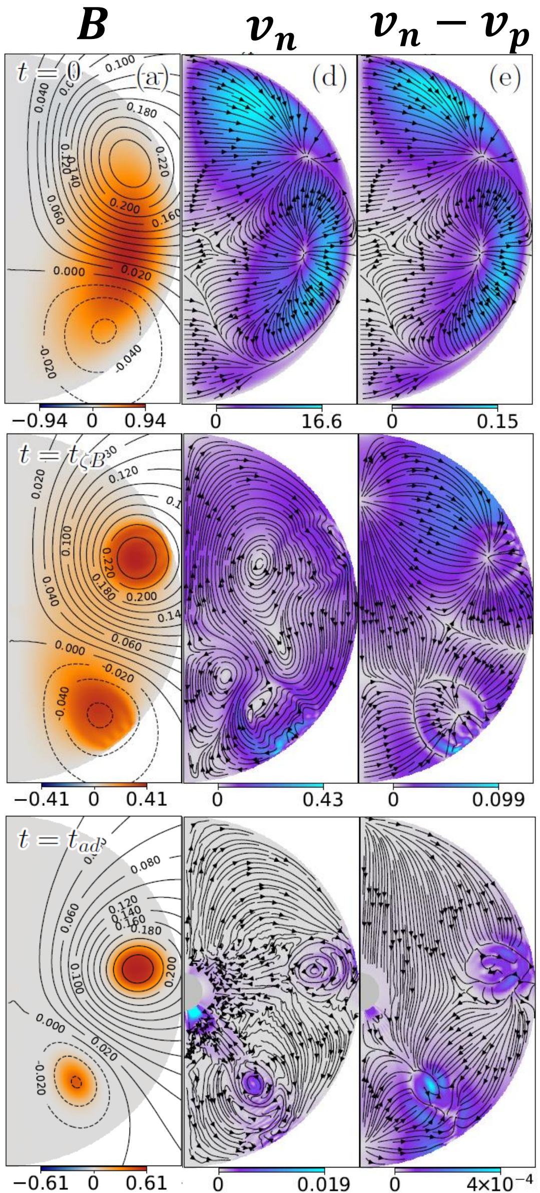
Conclusions

- **Operator representation of the magnetic field evolution in NS cores is developed for the simplest case**
- **The ambipolar diffusion operator $\hat{A}^{(p)}$ is linear, self-adjoint, its kernel = equilibrium fields, and it determines $\vec{B}$ evolution**
- **The key feature for evolution is $\mathbf{curl}\left(\vec{f}_L/n_c\right)$**

➢ **Many things to explore in the future work**

Thank you!

Simulations of Evolution



initial $\vec{B}$

quasistaionary evolution of $\vec{B}$

Grad-Shafranov equilibrium for $\vec{B}$

- Castillo, Reisenegger & Valdivia’20
- Moraga+’24

artificial friction

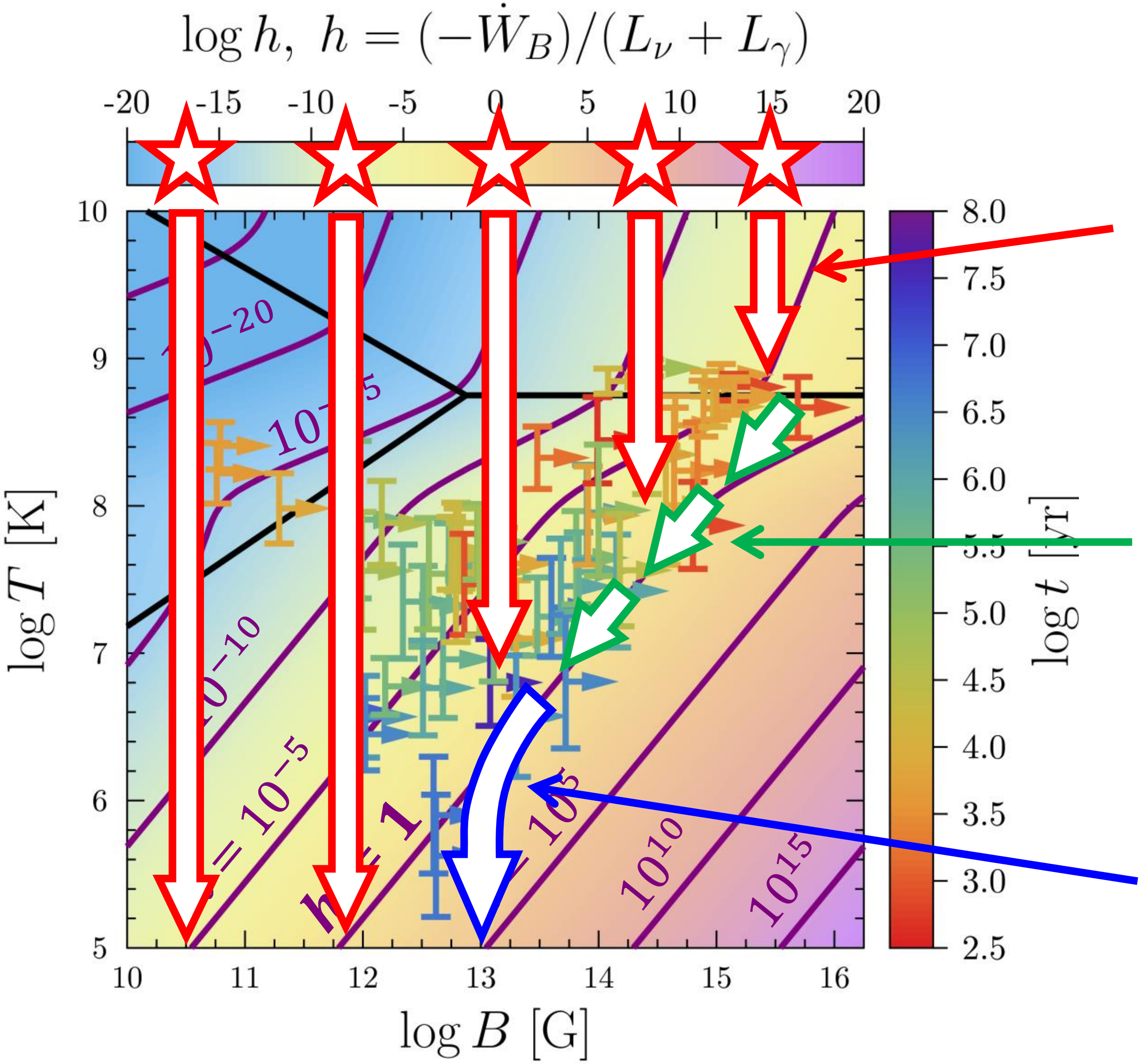
r.h.s. of Euler eqn. for n $\underbrace{+ (-\zeta\vec{u}_n)}_{\text{stabilize numerical scheme}}$

Troubles:

Realistic $\zeta \to 0$

Computational costs → ∞

Sketch of the Evolution



Non-magnetic cooling

Evolution along the heat balance line

Grad-Shafranov equilibrium for *B*

Quasistationary MHD + n, p Pairing

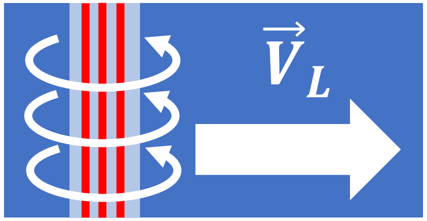
$T \ll T_{cp}, T_{cn}$ $\quad\quad$ $B < H_{c1}$

Lorenz force $\quad\quad$ Boyancy+Tension force

$$\frac{1}{4\pi}\mathrm{curl}\,\vec{B}\times\vec{B} \longrightarrow \frac{1}{4\pi}\mathrm{curl}\,\vec{H}_{c1}\times\vec{B}$$

Euler eqn. for n $\longrightarrow$ $\nabla\delta\mu_n = 0$

new "component": flux tubes = "Lines"



$e, \mu - p$ friction $\quad\quad$ $e, \mu - L$ friction

$$J_{ep}(\vec{u}_e - \vec{u}_p) \longrightarrow D_e(\vec{u}_e - \vec{V}_L)_\perp$$

$$\sum_{\ell=e,\mu} D_\ell(\vec{u}_\ell - \vec{V}_L)_\perp + en_p\frac{\vec{u}_p - \vec{V}_L}{c}\times\vec{B} + \frac{1}{4\pi}\mathrm{curl}\,\vec{H}_{c1}\times\vec{B} = 0$$

Total force balance on flux tubes

Gusakov, Kantor & DO 2020