Exploring universality with a many-body density functional

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Motivations

- Exploring systems from “**few**-body” to “**many**-body” within a unified picture

  consider a very powerful approach: Energy Density Functional
- However, mantain translation/Galileian invariances
- **here is a problem… but we will see how to overcome it**
- Study systems that are close to the unitary limit and are suited for effective expansion of the interaction

  we will see an example at the end

Summary

- Fast recall of Density Functional Theory (DFT) and Kohn-Sham (KS) equation

  (systems of interacting particles placed in an external one-body potential)
- Self bound systems and Hyperspherical Coordinates

  (interacting particles, no external one-body potential)
- Different formulation of DFT and KS equation

  (the many-body **hyperradial density**)
- Application to bosons close to the unitary limit (^{4}He atom clusters)

1: Fast recall of Density Functional Theory (DFT) and Kohn-Sham (KS) equation

(systems of interacting particles placed in an external one-body potential)

The EDF approach in a couple of slides:

P. Hohenberg and W. Kohn, Phys. Rev. 136, B864 (1964)

1) $E(\mathbf{n}) \gtrsim E_{gs}$ 2) $E(\mathbf{n}_{\mathbf{gs}}) = E_{gs}$

We have an Hamiltonian of interacting particles subject to an **external potential**

$$H = \sum_i^N \frac{p_i^2}{2m} + \sum_{i<j}^N V(\vec{r}_i - \vec{r}_j) + \sum_i^N v_{ext}(\vec{r}_i) \equiv \mathbf{T} + \mathbf{V} + \mathbf{v}_{ext}^{[1]}$$

$$E_{gs} = \langle \Psi_{gs} | T + V + \mathbf{v}^{[1]}_{\mathbf{ext}} | \Psi_{gs} \rangle$$

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$$E_{gs} = \langle \Psi_{gs} | \, T + V + \mathbf{v}^{[1]}_{ext} \, | \Psi_{gs} \rangle$$

$\mathbf{n} \equiv \mathbf{n}(\vec{\mathbf{r}})$ is the **one-body density**, namely the mean value of the one-body density operator $\sum_{i=1}^{N} \delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_i)$ on some N-body wave function

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$$\mathbf{n}(\vec{\mathbf{r}}) = \frac{1}{N} \int d\vec{r}_1 d\vec{r}_2 ... d\vec{r}_N \Psi^*(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i) \Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$$

And what is E(**n**) ? It is a particular functional of the one-body density defined as

$$E[\mathbf{n}] = \boxed{\langle \Psi^{\mathbf{n}} | T + V | \Psi^{\mathbf{n}} \rangle} + \int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}(\vec{r})$$

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$$\boxed{\langle \Psi^{\mathbf{n}} | T + V | \Psi^{\mathbf{n}} \rangle \equiv \min_{\Psi \to \mathbf{n}} \langle \Psi | T + V | \Psi \rangle} \equiv F(\mathbf{n})$$

The proof of the Theorem (following Levy 1979):

1) $E(\mathbf{n}) \geq E_{gs}$ Obvious! because of the Rayleigh–Ritz variational principle

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Proof of 2):

$$E[\mathbf{n}_{gs}] = F(\mathbf{n}_{gs}) + \int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r}) \geq E_{gs}$$

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therefore $$E_{gs} \geq F(\mathbf{n}_{gs}) + \int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r})$$

→ Equal!

The practical use of the theorem goes via the Kohn-Sham equation Phys. Rev. 140, A1133 (1965)

The **Kohn- Sham** **equation** is the Schroedinger equation of a **fictitious** system (the "Kohn-Sham system") of **independent** particles that generates the **same** n_{gs} **(r)** as any given system of **interacting** particles.

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$$H_{KS} = T + \sum_i W_{KS}(\vec{r}_i)$$

$$n^{KS} = n_{gs}$$

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$$H_{KS} = T + \sum_i W_{KS}(\vec{r}_i) \quad \Longleftrightarrow \quad n^{KS} = n_{gs}$$

Assuming the **W-representability of** $E(n)$, namely

$$E^W(n) = E(n)$$

→ solving the one-body **Kohn-Sham equation**

$$\left(-\frac{\nabla^2}{2m} + W_{KS}(\vec{r})\right)\psi_i(\vec{r}) = \epsilon_i \psi_i(\vec{r})$$

$$E^W(n_{gs}) = E(n_{gs}) = E_{gs}$$

By *reductio ad absurdum* one can show that

W_{KS} is unique!

But what is this one-body potential W_{KS} ???

At $n=n_{gs}$ E_{gs} is the minimum of E(n) namely

$dE^{V}(n)/dn = 0$ ⟶ $dT^{n^{V}}/dn + dV^{n}/dn + v_{ext}(r) = 0$

$dE^{W}(n)/dn = 0$ ⟶ $dT^{n,W}/dn + W(r) = 0$

=

At $n=n_{gs}$ E_{gs} is the minimum of E(n) namely

$dE^{V}(n)/dn = 0$ ⟹ $dT^{n,W}/dn + dT^{n,V}/dn - dT^{n,W}/dn + dV^{n}/dn + v_{ext}(r) = 0$

$dE^{W}(n)/dn = 0$ ⟹ $dT^{n,W}/dn + W(r) = 0$

=

At $n=n_{gs}$ E_{gs} is the minimum of E(n) namely

$$dE^{V}(n)/dn = 0 \Rightarrow dT^{n,W}/dn + \boxed{dT^{n,V}/dn - dT^{n,W}/dn + dV^{n}/dn + v_{ext}(r)} = 0$$

$$dE^{W}(n)/dn = 0 \Rightarrow dT^{n,W}/dn + \boxed{W(r)} = 0$$

Formally: $\mathbf{W(r) = dT^{n,V}/dn - dT^{n,W}/dn + dV^n/dn + v_{ext}(r)}$

As to $\mathbf{V^n(n)}$:

$$V(\mathbf{n}) \simeq U_H(\mathbf{n}) + V_{exc}(\mathbf{n}) + V_{corr}(\mathbf{n})$$

???

Moreover

$\mathbf{dT^{n,V}/dn - dT^{n,W}/dn}$

???

The KS Hamiltonian is not translation/Galileian invariant

(as is not the original Hamiltonian that contains an external field)

So, what to do for self bound systems ??

2: Self bound systems and Hyperspherical Coordinates

(interacting particles, no external one-body potential)

For self-bound systems one requires Translation / Galieian invariance

$$[H, \mathbf{P}_{CM}] = 0 \; / \; [H, \mathbf{R}_{CM}] = 0$$

$$H = \sum_i^N \frac{p_i^2}{2m} + \sum_{i<j}^N V(\vec{r}_i - \vec{r}_j) + \sum_i^N v_{ext}(\vec{r}_i) \equiv \mathbf{T} + \mathbf{V} + V_{ext}^{[1]}$$

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$$\Rightarrow \quad H = \frac{P_{CM}^2}{2Nm} + \frac{1}{2Nm} \sum_{i<j=1}^N |\vec{p}_i - \vec{p}_j|^2 + \sum_{i<j}^N V(\vec{r}_i - \vec{r}_j)$$

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$$\rightarrow \quad H = \frac{P_{CM}^2}{2Nm} + \frac{1}{2Nm} \sum_{i<j=1}^N |\vec{p}_i - \vec{p}_j|^2 + \sum_{i<j}^N V(\vec{r}_i - \vec{r}_j)$$

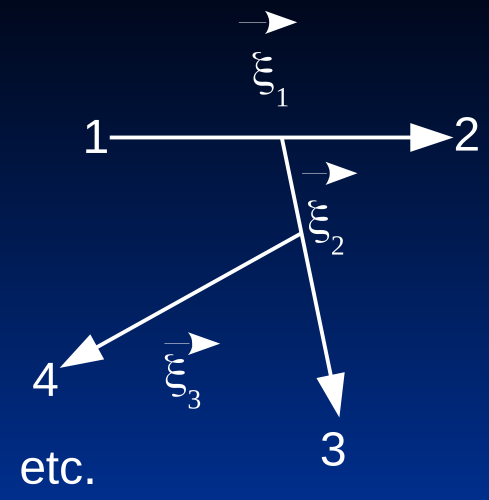
Invariant H_{inv}

Having eliminated the CM coordinate we need a set of N-1 vectors i.e. 3N-3 independent coordinates:

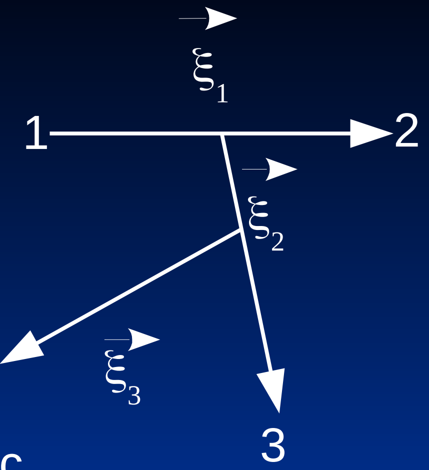
Jacobi coordinates

Jacobi coordinates

$\vec{\xi}_i$ = distances between each particle "i" and the cm of the previous (N – i) particles



Jacobi coordinates



etc.

$\vec{\xi}_i$ = distances between each particle "i" and the cm of the previous (N – i) particles

$$\begin{matrix} \vec{r}_1 & \vec{p}_1 \\ \vec{r}_2 & \vec{p}_2 \\ . & . \\ . & . \\ . & . \\ \vec{r}_N & \vec{p}_N \end{matrix} \quad \xleftrightarrow{\text{transformation}} \quad \begin{matrix} \vec{\xi}_0 = \vec{R}_{CM} & \vec{\pi}_0 = \vec{P}_{CM} \\ \vec{\xi}_1 & \vec{\pi}_1 \\ . & . \\ . & . \\ . & . \\ \vec{\xi}_{N-1} & \vec{\pi}_{N-1} \end{matrix}$$

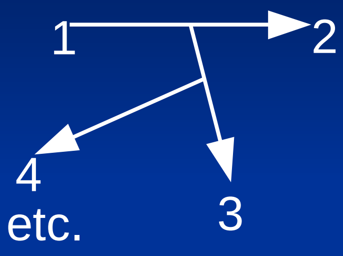
Remarks:

- When expressed in terms of Jacobi coordinates, any 1-body or 2-body potential becomes of "N-body nature"

- The translation invariant wave function is highly *correlated* (i.e. particles are not independent) beyond the correlation due to the dynamics

One can further transform the Jacobi coordinates into a new set of coordinates called Hyperspherical Coordinates

HYPERSPHERICAL COORDINATES



ξ_1
θ_1
ϕ_1
ξ_2
θ_2
ϕ_2
.
.
.
ξ_{N-1}
θ_{N-1}
ϕ_{N-1}

Total: 3N-3 coordinates

transformation →

ρ ← **hyperradius**
θ_1
ϕ_1
α_1
θ_2
ϕ_2
.
.
.
α_{N-2}
θ_{N-1}
ϕ_{N-1}

hyperangles (pointing to α_1 … α_{N-2})

Total: 3N-3 coordinates

HYPERSPHERICAL COORDINATES

ξ_1
θ_1
ϕ_1
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θ_2
ϕ_2
.
.
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Total: 3N-3 coordinates

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θ_1
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α_1
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ϕ_2
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.
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α_{N-2}
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ϕ_{N-1}

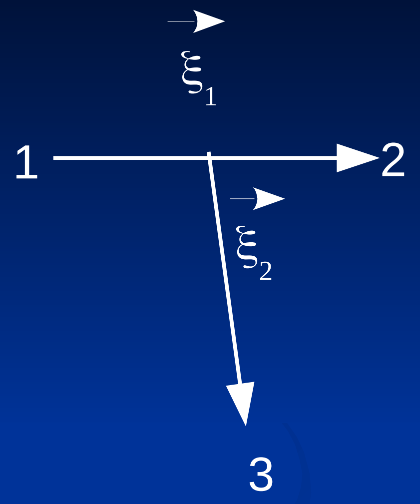
hyperangles

Total: 3N-3 coordinates

3N-4 angles are indicated with Ω

Total: 3N-3 coordinates: $(\rho \;\; \Omega)$

HOW ARE HYPERRADIUS ρ AND HYPERANGLES α_ι DEFINED ???

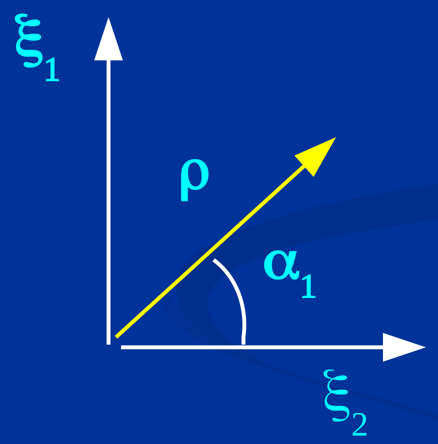


e.g. for **3** particles

ξ_1
θ_1
ϕ_1
ξ_2
θ_2
ϕ_2

transform →

$$\rho = \sqrt{\xi_1^{\,2} + \xi_2^{\,2}}$$
θ_1
ϕ_1
$$\alpha_1 = \mathrm{arcos}(\xi_2/\rho)$$
θ_2
ϕ_2



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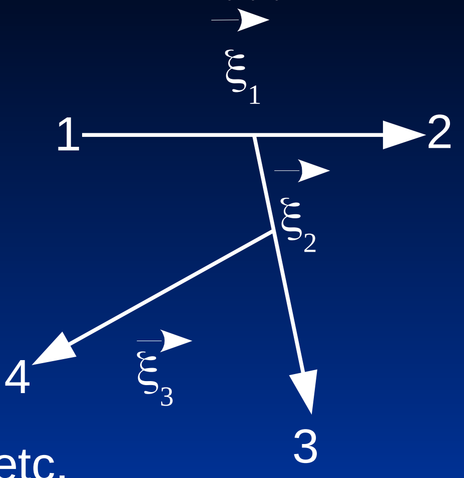
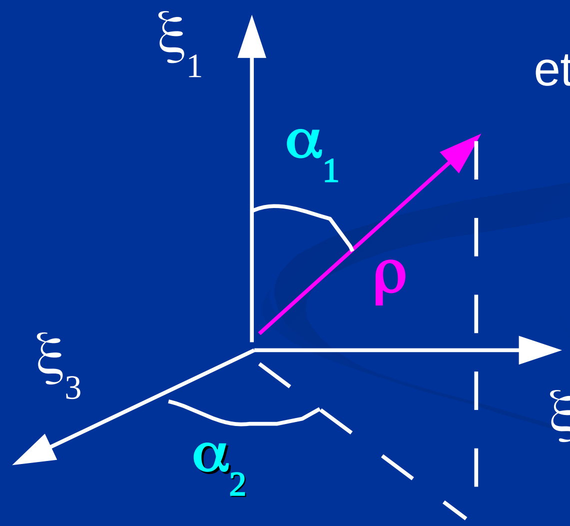


e.g. for **4** particles

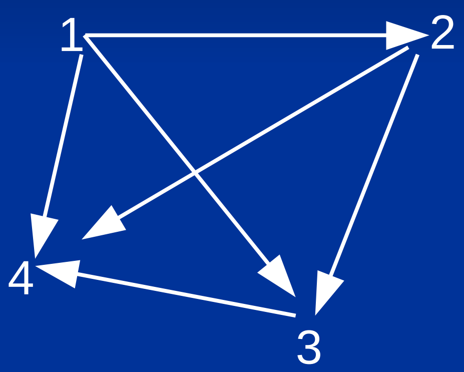
ξ_1		$\rho = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$
θ_1		θ_1
ϕ_1		ϕ_1
ξ_2	transformation →	α_1
θ_2		θ_2
ϕ_2		ϕ_2
ξ_3		α_2
θ_3		θ_3
ϕ_3		ϕ_3

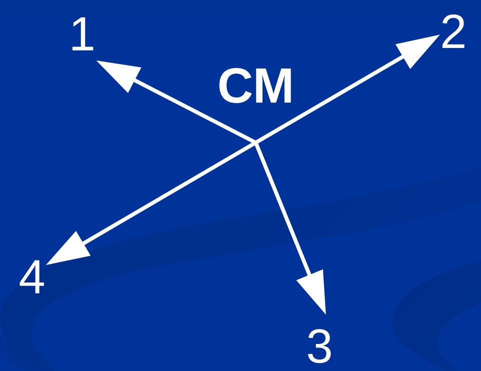


etc.

LET’S FOCUS ON THE HYPERRADIUS ρ :

$$\rho^2 \sim \Sigma_{ij} (\vec{r}_i - \vec{r}_j)^2 \qquad \rho^2 \sim \Sigma_i (\vec{r}_i - \vec{R}_{CM})^2$$





ρ can be onsidered as a highly “collective” variable

Very interesting feature of Hyperspherical Coordinates (HC):

With HC the expression of the 2 body invariant kinetic energy expressed in spherical coordinates is generalized to the N-body case

2 body: Kinetic Energy in SPHERICAL coordinates

$$T = \Delta_r - L^2/r^2 = -1/(2m)\,(\partial^2/\partial r^2 + 2/r\;\partial/\partial r) + L^2/r^2$$

The **spherical harmonics** $Y_{lm}(\theta, \phi)$ are the eigenfunctions of the **angular momentum** L^2

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N body: Kinetic Energy in HYPERSPHERICAL coordinates

$$T = \Delta_\rho - K^2/\rho^2 = -1/(2m)\,(\partial^2/\partial\rho^2 + (3N-4)/\rho\ \partial/\partial\rho) + K^2/\rho^2$$

The **hyperspherical** harmonics $Y_{K\ldots}(\Omega)$ are the eigenfunctions of **hyperangular momentum** K^2

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2 body: **SPHERICAL HARMONICS**

$$T = \Delta_r - L^2/r^2 = -1/(2m)\,(\partial^2/\partial r^2 + 2/r\ \partial/\partial r) + L^2/r^2$$

$$L^2\, Y_{lm}(\theta, \phi) = L\,(L+1)\, Y_{lm}(\theta, \phi)$$

N body: **HYPERSPHERICAL HARMONICS**

$$T = \Delta_\rho - K^2/\rho^2 = -1/(2m)\,(\partial^2/\partial\rho^2 + (3N-4)/\rho\ \partial/\partial\rho) + K^2/\rho^2$$

$$K^2\, Y_{K\ldots}(\Omega) = K\,(K+3N-5)\, Y_{K\ldots}(\Omega)$$

In terms of Hyperspherical coordinates the invariant Hamiltonian becomes

$$\Omega$$

$$H_{inv} = (\Delta_{\rho} - K^2/\rho^2) + V(\rho, \theta_1\phi_1 \ \theta_2\phi_2 \dots \alpha_1 \alpha_2 \dots)$$

$$= (\Delta_{\rho} - K^2/\rho^2) + V(\rho, \Omega)$$

Remark:

- When expressed in terms of Jacobi coordinates, even a 1-body operator becomes of “N-body nature”

Remarks in view of EDF:

- In H_{inv} there is no "real" one-body (IPM) density
- But one may define an analogous "many-body" density

$$n(r) \longrightarrow \nu(\rho)$$

$$r^2 n^{[1]}(r) = \int d\Omega_r\, d\vec{r}_2 ... d\vec{r}_N \Psi^*(\vec{r}, \vec{r}_2, ..., \vec{r}_N)\Psi(\vec{r}, \vec{r}_2, ..., \vec{r}_N) \longrightarrow \rho^{3N-4}\,\nu(\rho) = \int d\Omega\, \Psi^*(\rho, \Omega)\Psi(\rho, \Omega)$$

The idea is to try an EDF approach for $\nu(\rho)$

3: Different formulation of DFT and KS equation

(the many-body hyperradial density)

The EDF approach for $\nu(\rho)$

The **ANALOGOUS** of the Hohenberg Kohn statement:

1) $E(\nu) \gtrsim E_{gs}$ 2) $E(\nu_{gs}) = E_{gs}$

The EDF approach for $\nu(\rho)$

The **ANALOGOUS** of the Hohenberg Kohn statement:

1) $E(\nu) \gtrsim E_{gs}$ 2) $E(\nu_{gs}) = E_{gs}$

Given the invariant H $\quad \boldsymbol{H_{inv}} = (\Delta_\rho - K^2/\rho^2) + V(\rho, \Omega)$

What is E(ν) ?

$$E[\nu] = \langle \Psi^\nu | T + V | \Psi^\nu \rangle \equiv \min_{\Psi \to \nu} \langle \Psi | T + V | \Psi \rangle$$

The proof goes along the same line as before….

Before:

The proof of the Theorem (following Levy 1979):

1) $E(\mathbf{n}) \geq E_{gs}$ Obvious! because of the Rayleigh–Ritz variational principle

2) $E(\mathbf{n}_{gs}) = E_{gs}$

Proof of 2):

$$E[\mathbf{n}_{gs}] = F(\mathbf{n}_{gs}) + \int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r}) \geq E_{gs}$$ because of 1)

$$F(\mathbf{n}_{gs}) \equiv \min_{\Psi \to \mathbf{n}_{gs}} \langle \Psi | T + V | \Psi \rangle \leq \langle \Psi_{gs} | T + V | \Psi_{gs} \rangle$$ because it is a minimum

by definition $$E_{gs} = \langle \Psi_{gs} | T + V | \Psi_{gs} \rangle + \int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r})$$

therefore $$E_{gs} \geq F(\mathbf{n}_{gs}) + \int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r})$$

→ Equal!

Now:

The proof of the Theorem (following Levy 1979):

1) $E(\nu) \geq E_{gs}$ Obvious! because of the Rayleigh–Ritz variational principle

2) $E(\nu_{gs}) = E_{gs}$

Proof of 2):

$$E[\mathbf{n}_{gs}] = F(\mathbf{n}_{gs}) + \cancel{\int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r})} \geq E_{gs}$$ because of 1)

$\mathbf{n} \longrightarrow \nu$

$$F(\mathbf{n}_{gs}) \equiv \min_{\Psi \to \mathbf{n}_{gs}} \langle \Psi | T + V | \Psi \rangle \leq \langle \Psi_{gs} | T + V | \Psi_{gs} \rangle$$ because it is a minimum

by definition $$E_{gs} = \langle \Psi_{gs} | T + V | \Psi_{gs} \rangle + \cancel{\int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r})}$$

therefore $$E_{gs} \geq F(\mathbf{n}_{gs}) + \cancel{\int d\vec{r}\, v_{ext}(\vec{r})\, n^{[1]}_{gs}(\vec{r})}$$

⇒ Equal!

The practical use of the theorem goes via the "Analogous" of the Kohn-Sham equation

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...provided the **W-representability** of the functional $E(\nu)$

By *reductio ad absurdum* one can show that

W_{KS} is unique!

[redacted] One assumes that two hypercentral potentials, $W_1(\rho)$ and $W_2(\rho)$, differing by more than a constant, exist in such a way that the two Hamiltonians $H_1^W = T + W_1(\rho)$ and $H_2^W = T + W_2(\rho)$ have the same $\nu(\rho)$. Let us call $|\Phi_1\rangle$ and $|\Phi_2\rangle$ the respective wave functions and $\mathcal{E}_1$ and $\mathcal{E}_2$ the corresponding energies. From the Rayleigh-Ritz variational principle the following condition holds:

$$\mathcal{E}_1 < \langle\Phi_2|H_1^W|\Phi_2\rangle = \langle\Phi_2|H_2^W|\Phi_2\rangle + \langle\Phi_2|H_1^W - H_2^W|\Phi_2\rangle, \tag{28}$$

$$\mathcal{E}_1 < \mathcal{E}_2 + \int d\rho\, \rho^{3(N-4)}\, [W_1(\rho) - W_2(\rho)]\, \nu(\rho). \tag{29}$$

The same can be repeated starting from $\mathcal{E}_2$ arriving at

$$\mathcal{E}_2 < \mathcal{E}_1 + \int d\rho\, \rho^{3(N-4)} [W_2(\rho) - W_1(\rho)]\, \nu(\rho). \tag{30}$$

Summing both inequalities we arrive at the following contradiction, $\mathcal{E}_1 + \mathcal{E}_2 < \mathcal{E}_1 + \mathcal{E}_2$, proving that the first assumption was wrong. Accordingly, it is proven that the density $\nu(\rho)$ uniquely determines the hyper-radial potential $W(\rho)$ that generates it. [redacted]

The practical use of the theorem goes via the "Analogous" of the Kohn-Sham equation

$$[\, \Delta_\rho + K^2/\rho^2 + W_{AKS}(\rho) \,]\, \Phi_{[K\mathrm{min}]}(\rho) = E_{gs}\, \Phi_{[K\mathrm{min}]}(\rho)$$

$$\rho^{(3N-4)}\, \nu_W(\rho_{gs}) = |\Phi_{[K\mathrm{min}]}(\rho)|^2$$

$K_{min} = 0$ for bosons $K_{min} \neq 0$ for fermions

for KS: $W_{KS}(\rho)$???

---

for AKS: $W_{AKS}(\rho)$???

At $\nu=\nu_{gs}$ E_{gs} is the minimum of $E(\nu)$ namely

$dE^{V}(\nu)/d\nu = 0$ ⇨ $dT^{nV}/dn + dV^{n}/dn = 0$

$dE^{W}(\nu)/d\nu = 0$ ⇨ $dT^{n,W}/d\nu + W(\rho) = 0$

=

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At $\nu = \nu_{gs}$ E_{gs} is the minimum of $E(\nu)$ namely

$$dE^{V}(\nu)/d\nu = 0 \Rightarrow dT^{\nu,W}/d\nu + \boxed{dT^{\nu,V}/dn - dT^{\nu,W}/d\nu + dV^{\nu}/d\nu} = 0$$

$$dE^{W}(\nu)/d\nu = 0 \Rightarrow dT^{\nu,W}/d\nu + \boxed{W(\rho)} = 0$$

=

Simplest guess:

remember

$$H_{inv} = (\Delta_\rho - K^2/\rho^2) + V(\rho, \Omega)$$

$$= V^{[2]}(\rho, \Omega) + V^{[3]}(\rho, \Omega) + \ldots$$

Try integral on the hyperangular part of the ground state wave function

Sort of "mean field" for the ρ coordinate!

$$W_{AKS}(\rho) = N(N-1)/2 \int d\Omega\, V^{[2]}(\rho, \Omega)\, |Y_{[K_{min}]}(\Omega)|^2 + N(N-1)(N-2)/6 \int d\Omega\, V^{[3]}(\rho, \Omega)\, |Y_{[K_{min}]}(\Omega)|^2 + \ldots$$

4: Application to bosons close to the unitary limit (^{4}He atoms)

Helium clusters

Remarks:

The dimer of ^{4}He has a binding energy of about **1 mK**, three orders of magnitude less than the typical energy scale of $\hbar^2 / m \; r^2_{vdW}$ = **1.677 K**,

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The **first term** of this expansion is a **contact interaction** between the two helium atoms. However, as it is well known, the three-body system (as well as larger systems) collapses, even if the contact interaction is set to produce an infinitesimal binding energy. This phenomenon is known as the **Thomas collapse** and it is remedied by the introduction of a contact **three-body force** set to correctly describe the trimer energy

Accordingly, the **leading order (LO)** of this effective theory has two terms,

$$V_{LO}^{[2]} = \sum_{i<j} A e^{-r_{ij}^2/\alpha^2}, \quad V_{LO}^{[3]} = \sum_{i<j<k} B e^{-r_{ijk}^2/\beta^2},$$

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Several choices are possible for **B** and **β**, for exemple

a) *fit to **trimer** and **tetramer** binding energies*

b) *in view of the fact that W (ρ) has to account for energies at any N, one can obtain couples **(B, β) values**, all fitting the **tetramer** binding energy.*

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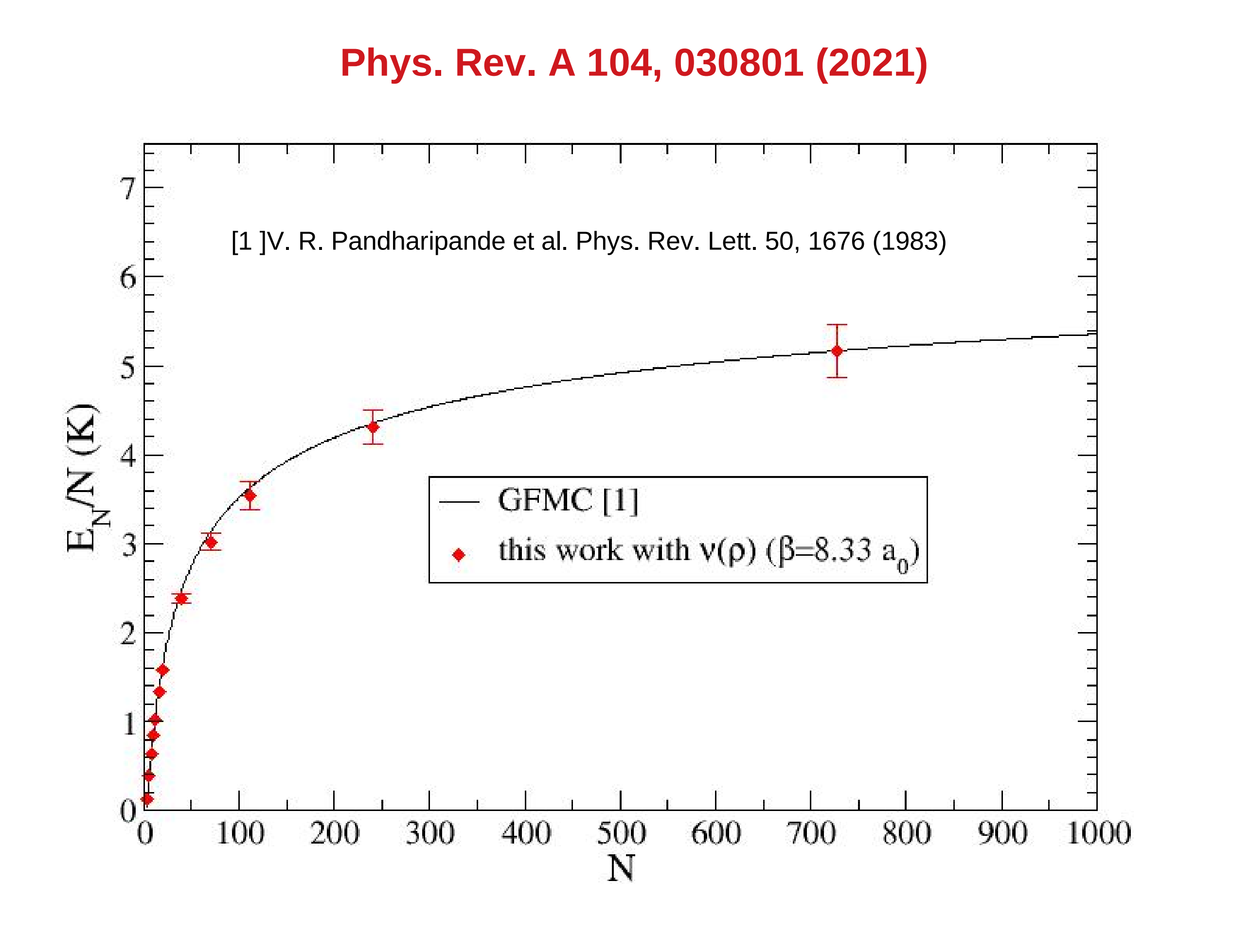
RESULTS FOR BINDING ENERGIES

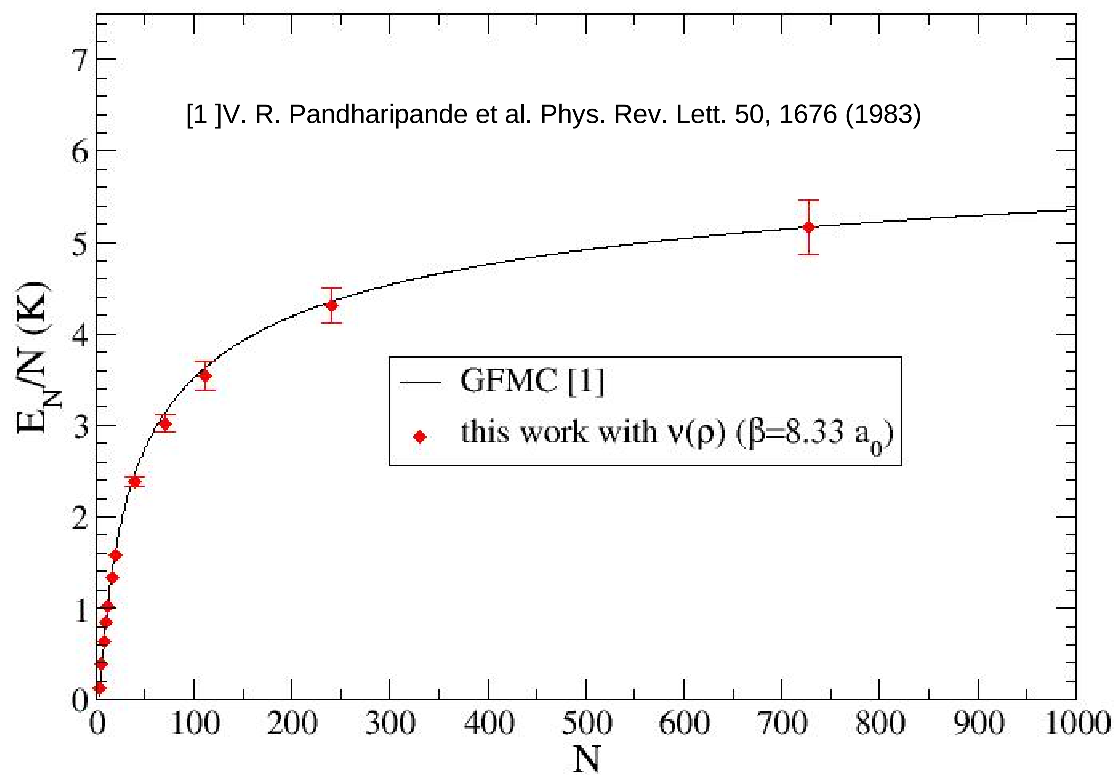
FOR ANY NUMBER N OF PARTICLES

Binding energy per particle for any N



Binding energy per particle for any N

Phys. Rev. A 104, 030801 (2021)



For the **lowest N** values we observe ***substantial independence*** from the three-body range **β** with the overall best description inside the interval **7.5 a 0 < β < 9.0 a** $_0$

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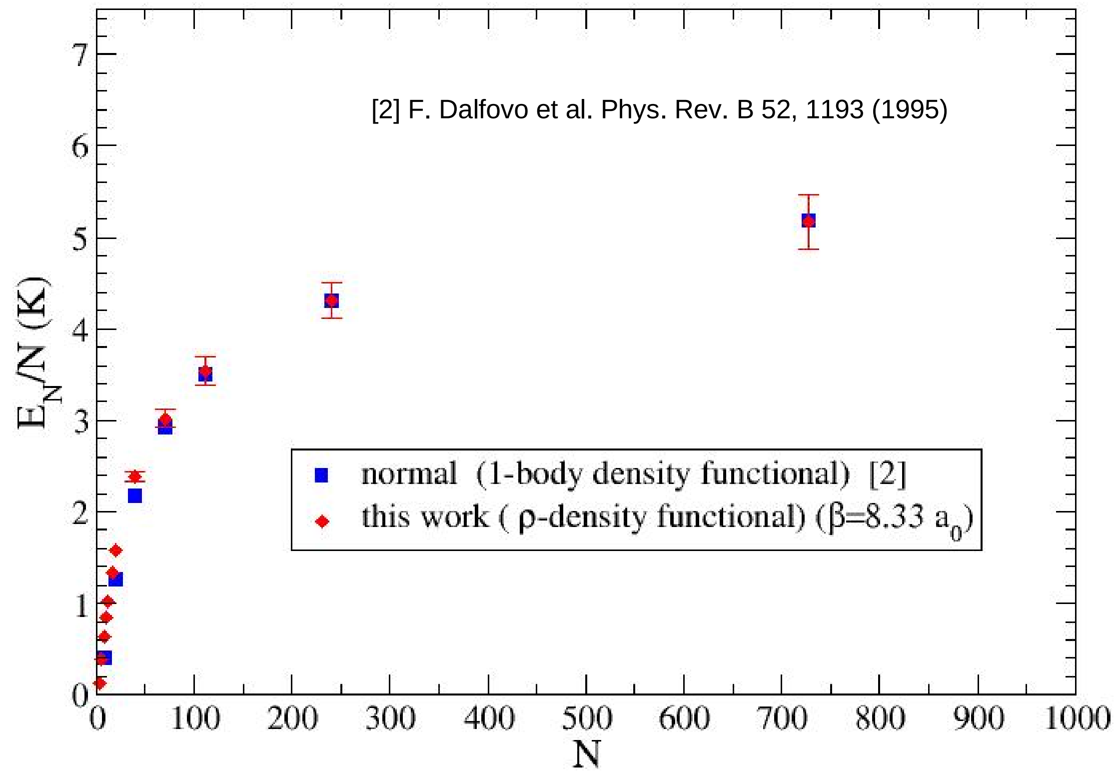

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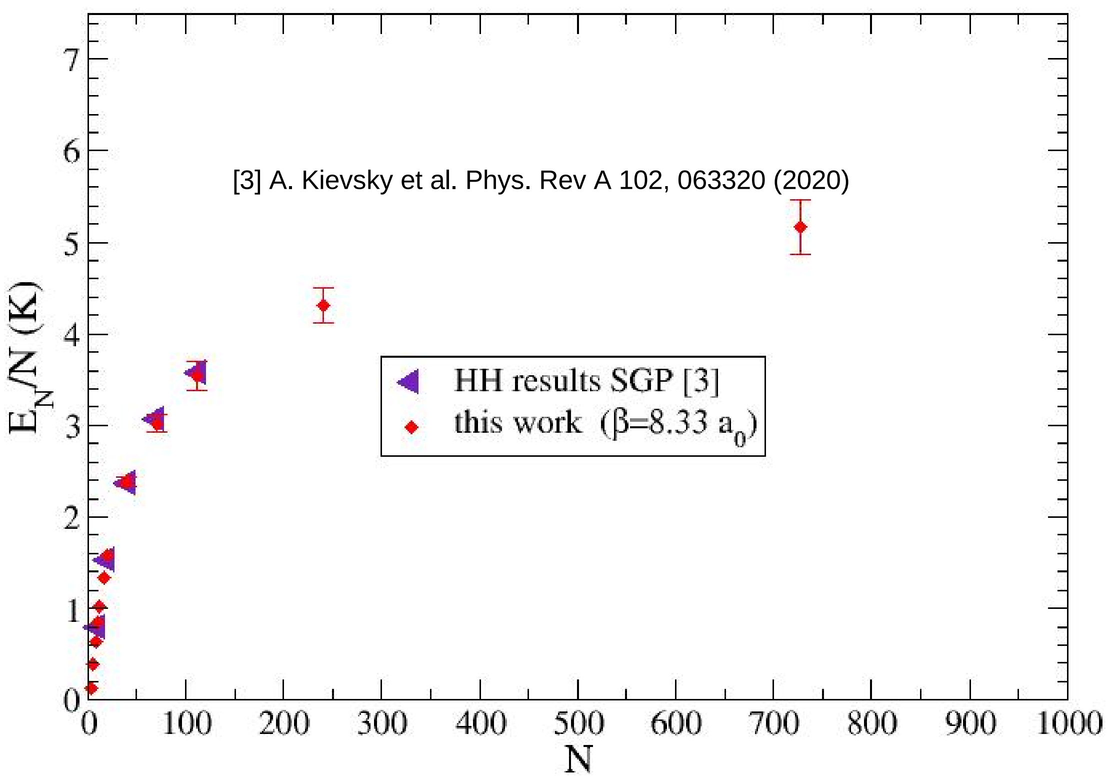

(reduced) many-body density **$\nu(\rho)$** for selected number of particles

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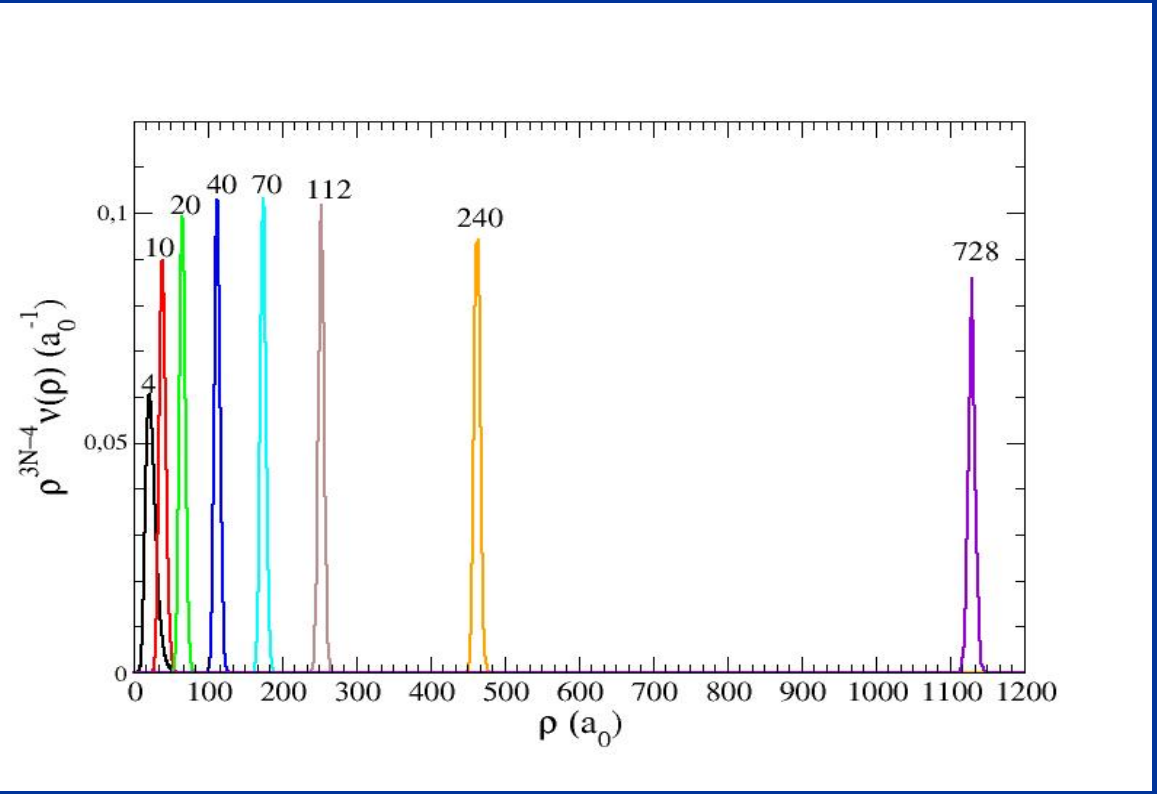
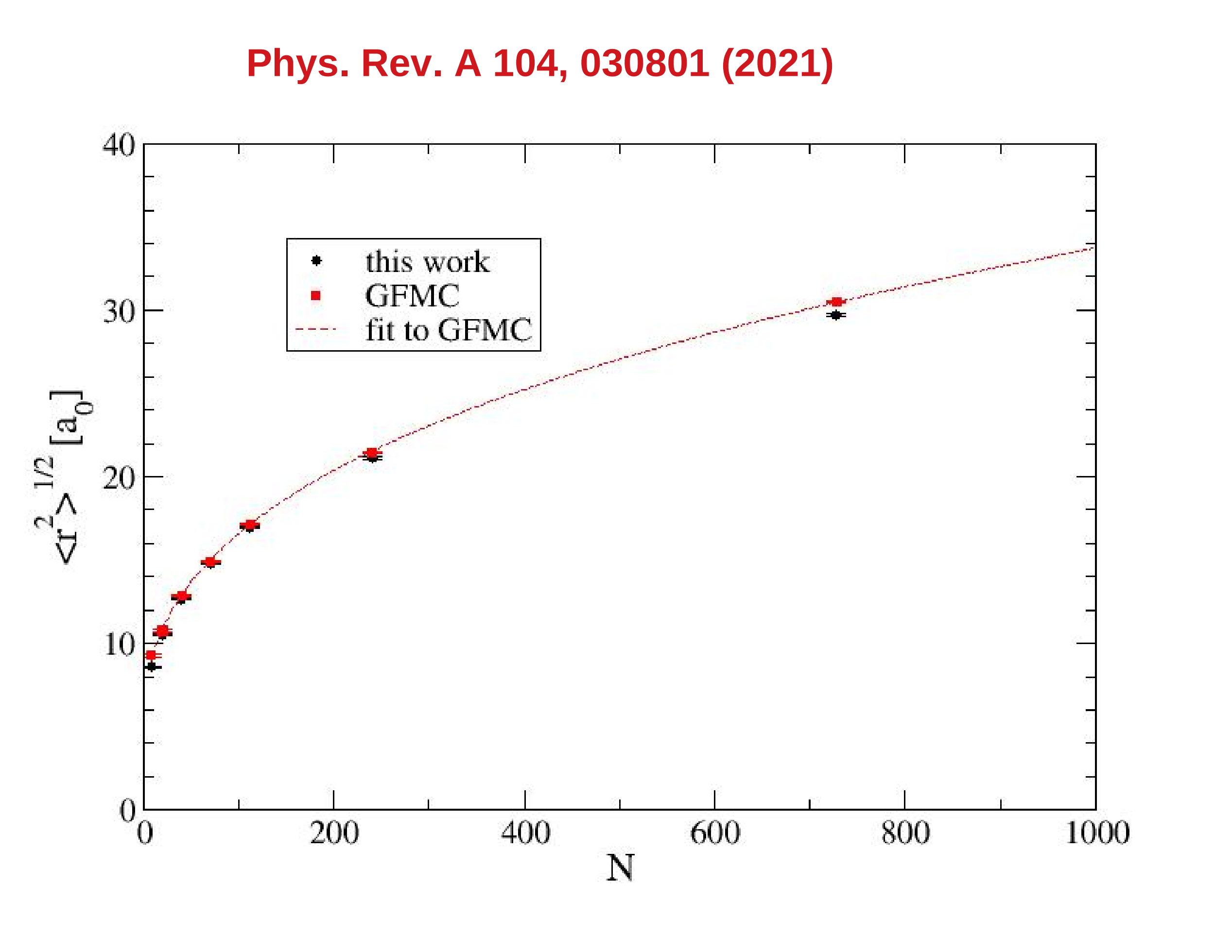


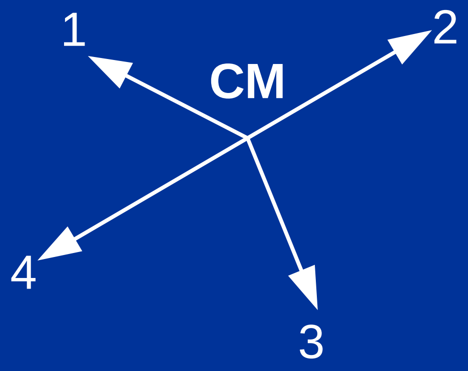
Extremely **localized density** around a value almost **linear with N** .

Very compact object. Closer particles are discouraged (incompressible?)
Also larger values are discouraged.

Mean square radius

$\rho^2 \sim \Sigma_i (\vec{r}_i - \vec{R}_{CM})^2$





CONCLUSIONS

- An energy density functional approach has been formulated in terms of the density **$\nu(\rho)$** where **ρ** is a translation invariant variable of collective nature
- It has been shown that the functional **$E[\nu]$** is governed by a **unique** (unknown) **hyperradial potential $W(\rho)$**.
- The solution of a **single hyperradial equation** with such an hyperradial potential allows to determine the **binding energy for any N** in a straightforward way.
- We have applied this framework to the bosonic case focusing on **^{4}He clusters**.
- The guess for **$W(\rho)$** has been **inspired by the effective theory** approach together with a **generalization of the mean field** concept.
- Extremely satisfying results have been found. The key point has been **using the range of the three-body interaction β**, to fine tune the **$W(\rho)$**.

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OUTLOOK

- Extension to **trapped systems**
- Extension to **Fermions.** In Nuclear Physics: **W (ρ)** ??? EFT ???

And much more to explore with the AKS equation and

the Many-Body Density Functional $E(\nu(\rho))$!!!