HOBET:

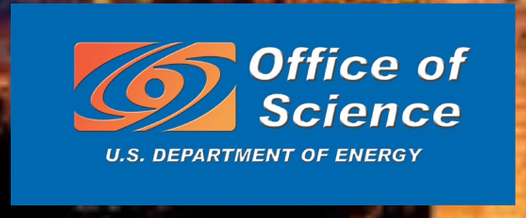
LEC Fitting, Counting, the Pion, and Light Nuclei

I. Physics issues

II. Operator structure, power counting, LEC fitting

III. Three- and A-body structure

Chiral EFT: New Perspectives INT Mach 17-21, 2025



expand the "included space"
to mitigate: no-core SM

reformulate the problem as
an NRET: HOBET

PHYSICAL REVIEW C VOLUME 52, NUMBER 5 NOVEMBER 1995

Large-basis shell model studies of light nuclei with a multivalued *G*-matrix effective interaction

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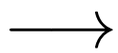
(Received 10 July 1995)

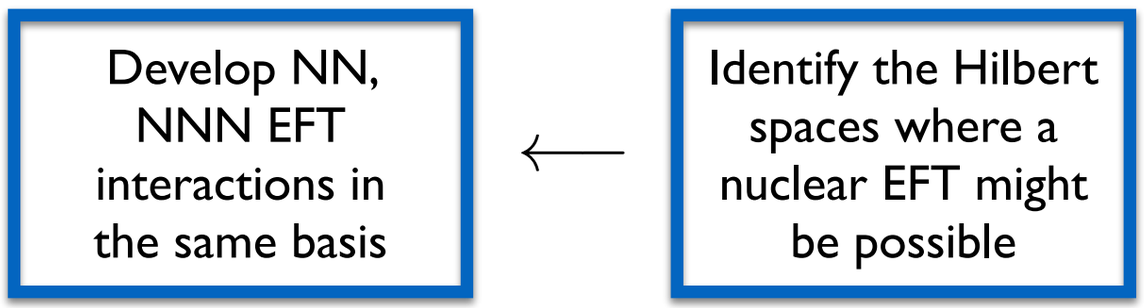
Large-basis shell model studies of low-lying excitations in light nuclei from ^{4}He to ^{7}Li have been performed with a multivalued G-matrix effective interaction, as recently suggested by Haxton *et al.* Calculations were performed relative to the vacuum ("no core") using very large, separable model spaces containing all excitations with unperturbed energies up to $8\hbar\Omega$. Using G matrices derived from a new Nijmegen potential, we achieve a very satisfactory description of these excitations.

Shell model not consistent with the necessary
form off the effective interaction

Develop NN,NNN interactions in a convenient basis



Hope one can find a MB formulation that requires only the retained info.



C. L. Song: HOBET as a potential theory, to better understand its properties

Tom Luu: The separation of the needed UV and IR integrations

WH: Abstracting the theory: HOBET without potentials

K. McElvain: LEC determination from phase shifts

KM+WH: A=3,4

HOBET Attributes

- Reduction of QCD directly to the nuclear scale
  - The reduction preserves translational invariance, continuity in energy
  - Rate of convergence can depend on the nuclear energy scale Λ_N and wave-packet parameter b, but results always agree to within respective convergence tolerances: there are no arbitrary cutoffs
  - Experimental information previously encoded in potentials (phases shifts) is instead used directly in fixing operator coefficients

All operators have meaning only within the low-energy/momentum Hilbert space and are regulated by Λ_N and b

Consequently the cutoff/regulator issues that become significant in chiral potentials at higher scales ($\sim$ 450 MeV) are absent

Tews, Huth, Schwenk PRC 98 (2018) 024001
Reinert, Krebs, Epelbaum Eur Phys J A 54 (2018) 86
Epelbaum, Krebs, Meissner, Eur Phys J A 51 (2015) 53

Included Space

- position basis (lattice) $\leftrightarrow$ harmonic oscillator $\leftrightarrow$ momentum basis
  - Wave packet bases are more efficient in describing bound states or reactions over the region where the strong potential operates
  - The HO is the unique choice, because of its translational invariance
  - The HO is position-momentum self conjugate: it provides a parameter in the ET that has physical relevance: r/b $\sim$ qb ~ 1
    - This parameter plays other roles: distinguishes the pion in HOBET from other interactions, and distinguishes singlet and triplet channels
- The HO is JUST an ET basis: scattering and bound states are treated in a unified and even-handed way
- (Λ_N,b) both define P and regulate the interactions introduced in P

- The operator expansion in HOBET is not one around r=0

  It is simultaneously an expansion in r around b
  and in p around 1/b

- The theory is forgiving in the choice of b, but good choices are those that make these two expansions equivalently effective - the proper physical choice

- This position/momentum equivalence is encoded in the HO ladder operators, so it is no surprise that these are the building blocks of HOBET's operators

Relation to Potential Theory

- HOBET's form is constrained by the requirement that it reduces to potential theory in the limit of local operators, so HOBET must reproduce potential predictions - thus making the theory testable at every step

$$H^{eff}P|\Psi\rangle = EP|\Psi\rangle \qquad H^{eff} = P\left[H + H\frac{1}{E-QH}QH\right]P$$

$$H \to \frac{1}{2}\sum_{i\neq j=1}^{A}[T_{ij}+V_{ij}]$$

- Properties of the BH equation:
  - must be solved self-consistently: E is the resulting eigenvalue (but easy)
  - applies equally to E<0 and E>0
  - there are an infinite number of solutions, though P is finite: every state with a nonzero overlap with P is generated
  - solutions yield the exact E and exact projections of the eigenvectors: nonorthogonal

- Might think at this point that an energy-dependent interaction is an unworkable starting point for an ET

- But in fact all the simplicity of theories with fixed LECs can be retained

- Once this is recognized, the explicit energy dependence becomes a major advantage

- HOBET's LECs are derived from physical data mapped in E, $\delta(E)$

- The explicit energy dependence allows one to precisely determine the normalization of BH solutions

- Elegant solution to the intruder state problem: there are none

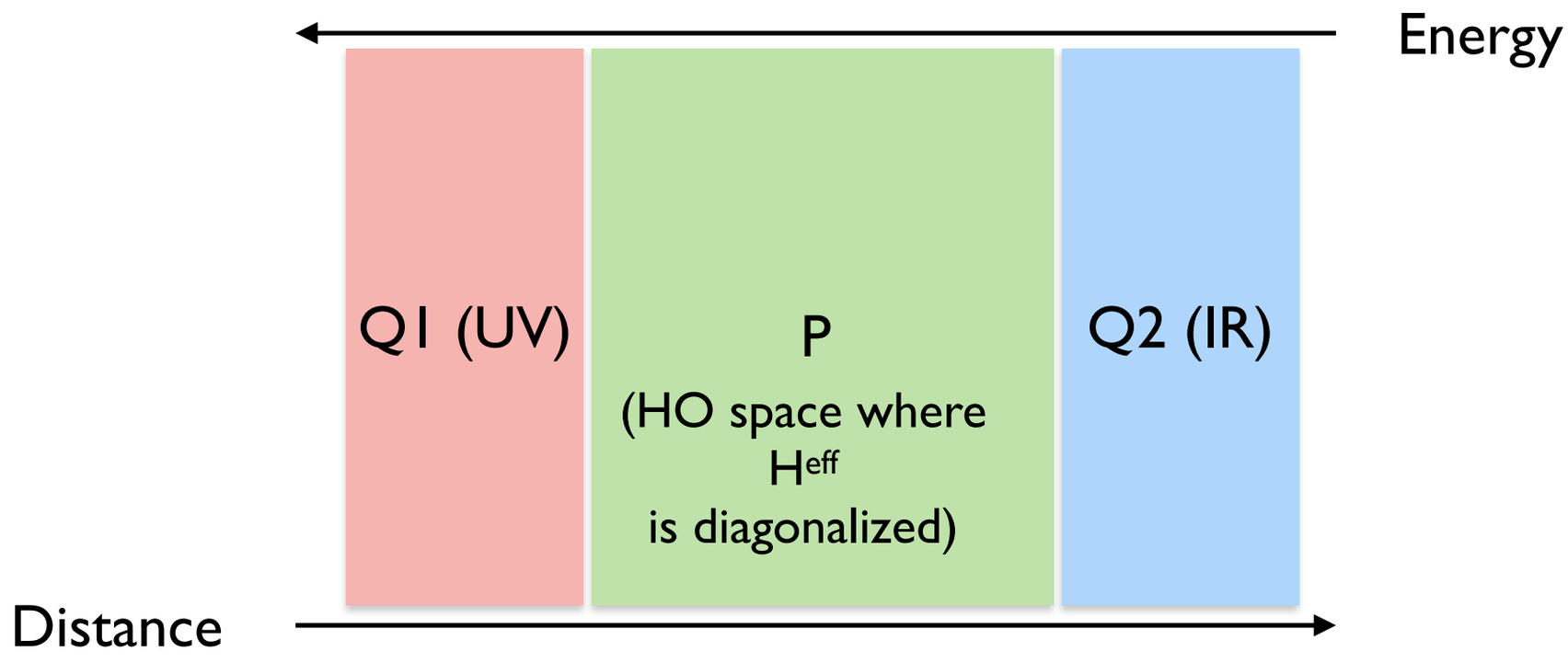
Schucan and Weidenmuller, Annal. Phys. 76 (1973) 483
Barrett and Kirson, NPA 148 (1970) 145

Formulating the ET

- HOBET's symmetric expansion of position and momentum around b and 1/b omits both IR and UV physics

- For optimal b, these omissions are of comparable importance

- But differ greatly in their consequences for the ET



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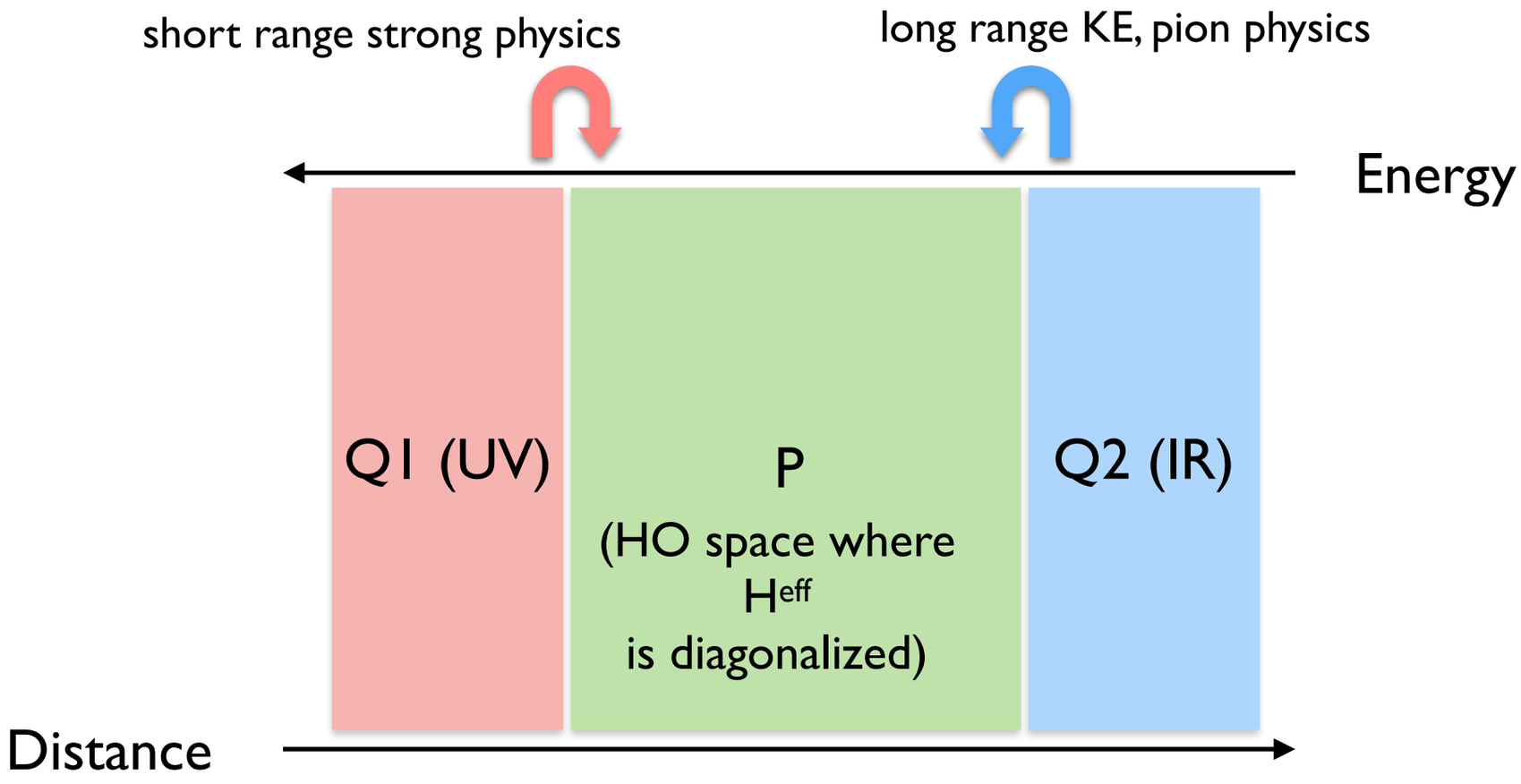

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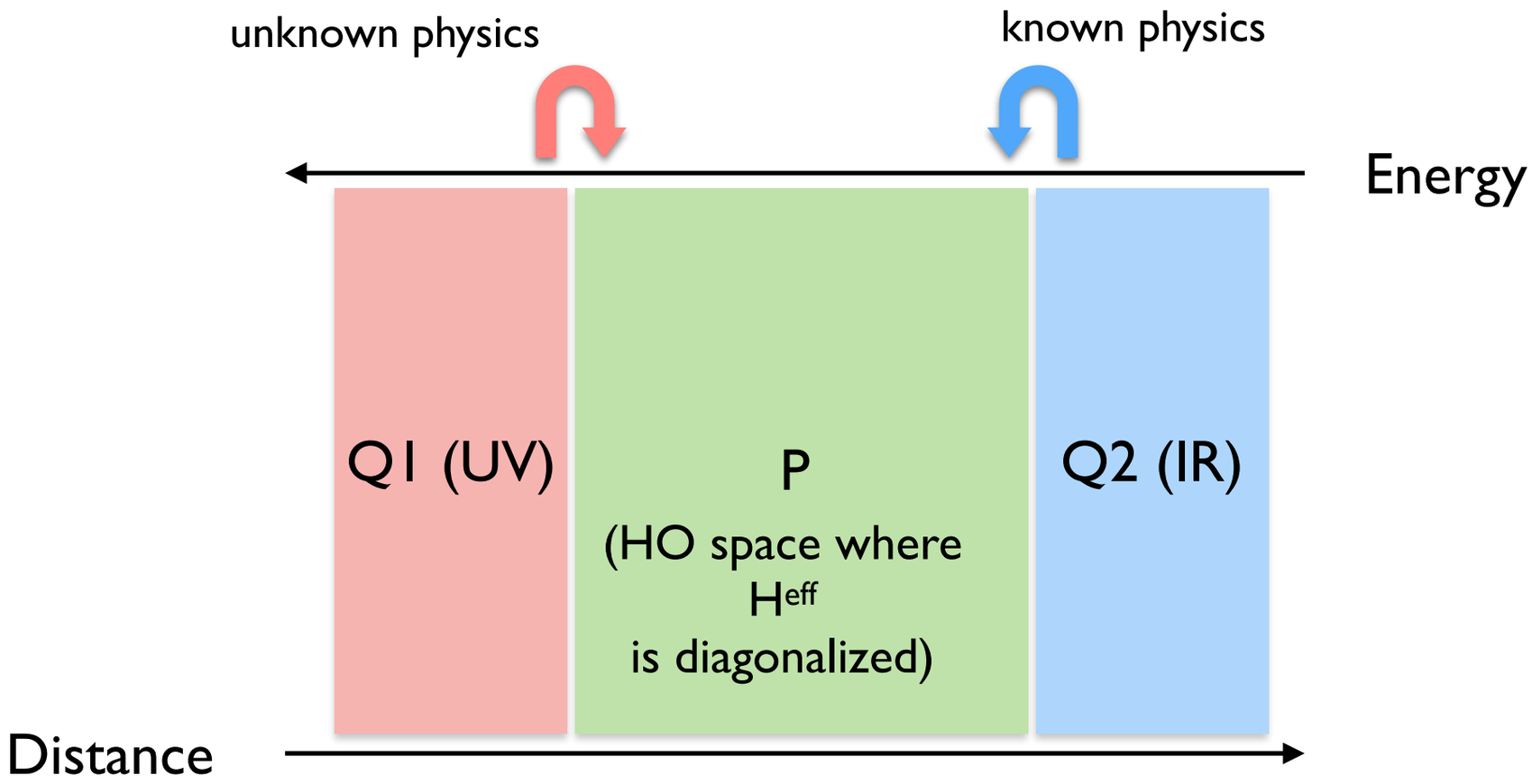

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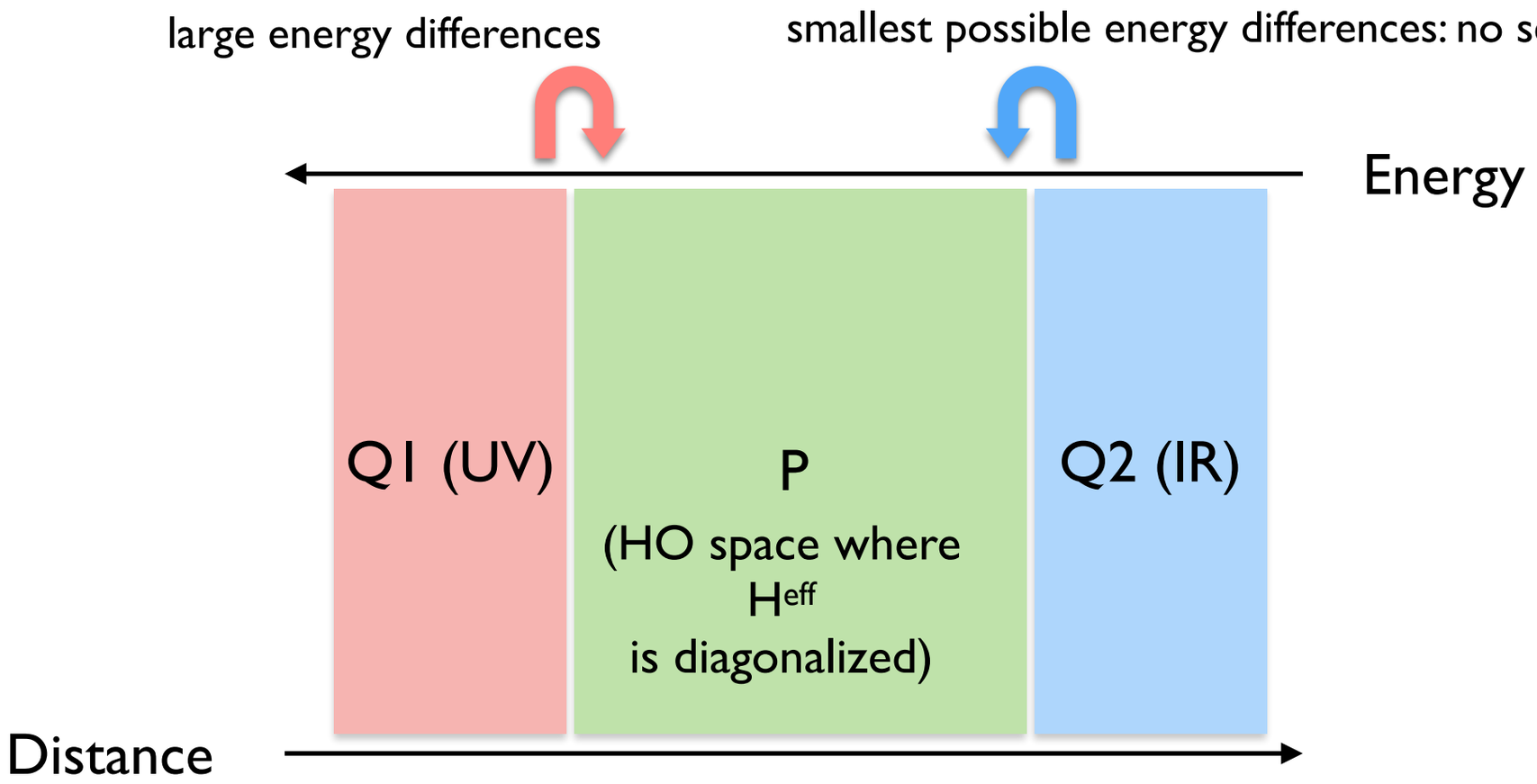

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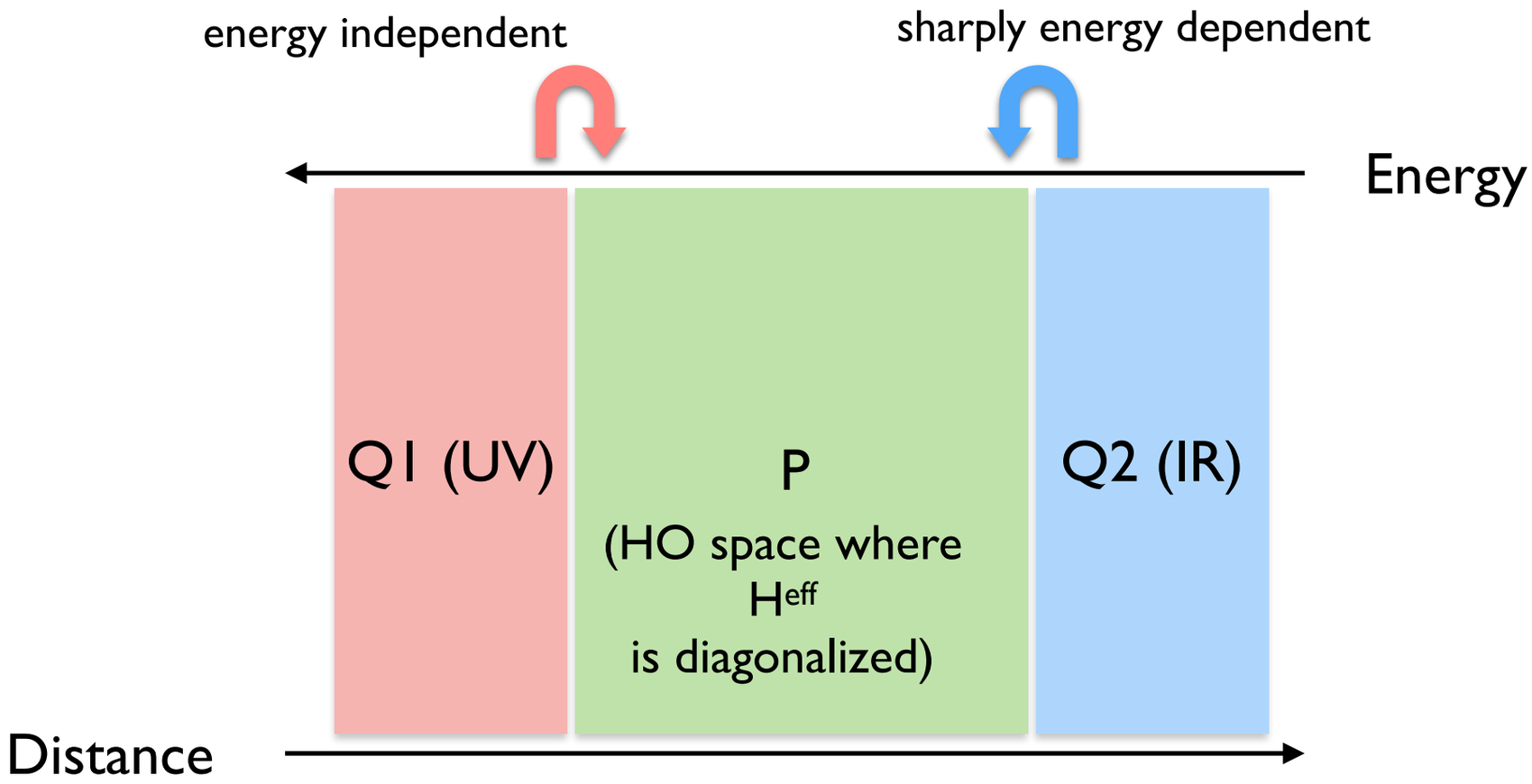

Re-sum the BH Equation to Separate the Two Contributions

$$H = H^{IR} + V^{UV}$$

$$G_{QH^{IR}} = \frac{E}{E - QH^{IR}} \qquad G_{QH} = \frac{1}{E - QH}$$

$$PH^{\text{eff}}P|\Psi\rangle = EP|\Psi\rangle$$

$$H^{eff} = H^{IR}G_{QH^{IR}} + G_{H^{IR}Q}\left[V^{UV} + V^{UV}\,G_{QH}\,QV^{UV}\right]G_{QH^{IR}}$$

WH and Tom Luu

- The re-summation is exact and when applied to a potential, cures slow convergence problems that Tom explored

Re-sum the BH Equation to Separate the Two Contributions

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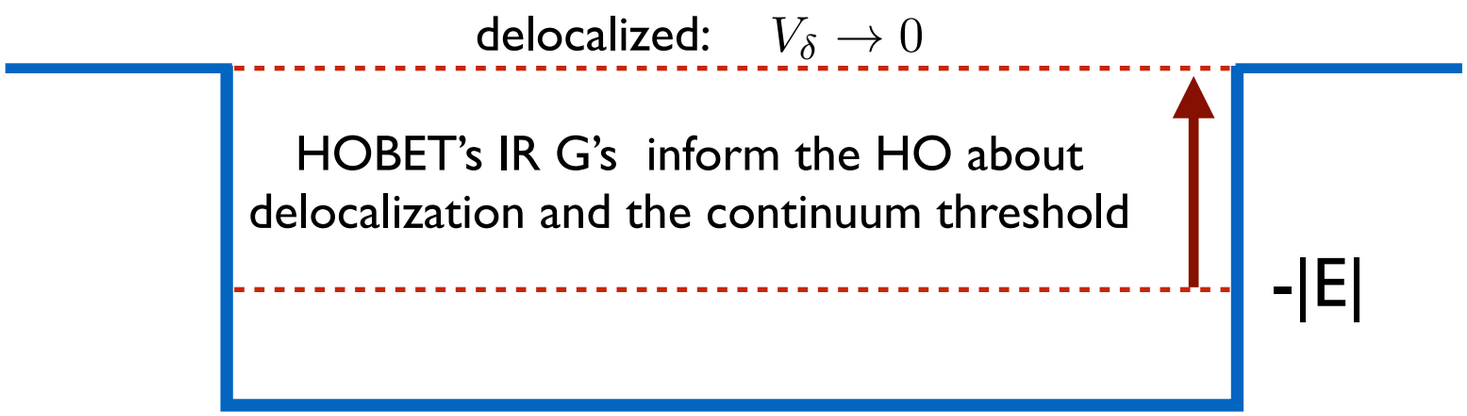
WH and Tom Luu

- 95% of the energy dependence is absorbed by the infra-red Green's functions that are relatively easy to calculate
- Any residual energy dependence is easily absorbed the momentum dependence of the operators in the expansion we will introduce shortly

This form reflects important physics: like many systems, nuclei involve a competition between the kinetic energy, which is minimized by delocalizing, and the potential energy, minimized by localization

The control parameter for this physics is the binding energy



Arises naturally and explicitly in HOBET, regardless of how extreme the weak binding becomes

Build the Effective Theory

$$H = H^{IR} + V^{UV}$$

$$G_{QH^{IR}} = \frac{E}{E - QH^{IR}} \qquad\qquad G_{QH} = \frac{1}{E - QH}$$

$$PH^{\text{eff}}P|\Psi\rangle = EP|\Psi\rangle$$

$$H^{eff} = H^{IR}G_{QH^{IR}} + G_{H^{IR}Q}\underbrace{\left[V^{UV} + V^{UV}\,G_{QH}\,QV^{UV}\right]}_{V_\delta}G_{QH^{IR}}$$

- The isolated short range physics is replaced by an operator expansion constructed in second quantization from the HO ladder operators, with the LECs as constants. So one has a simple theory but also the needed explicit mapping from state energy to the observable, $\delta(E)$

Resume the BH Equation to Separate the Two Contributions

$$H = H^{IR} + V^{UV}$$

$$G_{QH^{IR}} = \frac{E}{E - QH^{IR}} \qquad\qquad G_{QH} = \frac{1}{E - QH}$$

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- With the replacement all reference to short-range potentials is removed
- The information previously encoded in potentials must now be used directly to determine the LECs: LECs fixed by phase shifts

Ladder operator construction of V_δ

$$(a_x^\dagger, a_y^\dagger, a_z^\dagger): \qquad a_i \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r_i} + r_i\right) \qquad a_i \equiv \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial r_i} + r_i\right)$$

$$\boldsymbol{r} = \frac{1}{\sqrt{2}b}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad a_M^\dagger = \hat{e}_M \cdot \boldsymbol{a}^\dagger \qquad \tilde{a}_M = (-1)^M a_{-M}$$

- From these operators one constructs the nodal and angular momentum raising and lowering operators,

$$\tilde{\mathbf{a}} \odot \tilde{\mathbf{a}}\ |n\ell m\rangle = -2\ \sqrt{(n-1)\,(n+\ell-1/2)}\ |n-1\,\ell m\rangle$$

$$[[\tilde{\mathbf{a}} \otimes \tilde{\mathbf{a}} \otimes \cdots \otimes \tilde{\mathbf{a}}]_\ell \otimes |n\ell\rangle]_{00} = \ (-1)^\ell\, 2^{\ell/2} \sqrt{\frac{l!}{(2\ell-1)!!}\,\frac{\Gamma[n+\ell+\frac{1}{2}]}{\Gamma[n+\frac{1}{2}]}}\ |n00\rangle \quad (2)$$

from which V_δ is constructed

- Operators are defined in P and generated by a progression in oscillator quanta

$$V_\delta^S = a_{LO}^{\rm S}\delta(\boldsymbol{r}) + a_{NLO}^{\rm S}\left(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger\ \delta(\boldsymbol{r}) + \delta(\boldsymbol{r})\ \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}}\right) + \cdots$$

$$\delta(\mathbf{r}) = \sum_{n'n\ \in P} |n'00\rangle\langle n'00|\delta(\mathbf{r})|n00\rangle\langle n00| = \sum_{n'n\ \in P} d^{00}_{n'n}|n'00\rangle\langle n00| \qquad \text{(Regulated by P)}$$

$$d^{00}_{n'n} \equiv \frac{2}{\pi^2}\left[\frac{\Gamma(n'+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(n'-1)!(n-1)!}\right]^{1/2}$$

(nodal quantum no. expansion)

$$\langle n'(\ell'=0S)JM;TM_T|V_\delta^{\rm S}|n(\ell=0S)JM;TM_T\rangle = d^{00}_{n'n}\left[a_{LO} - 2[(n'-1)+(n-1)]a_{NLO}^{\rm S} + \cdots\right]$$

- If we had computed the LECs from a potential, we would have found that the LECs are a non-local generalization of the familiar Talmi integrals

$$\int d\boldsymbol{r}'d\boldsymbol{r}\ r^{2p'}e^{-r'^2/2}Y_{00}(\Omega')\,V(\boldsymbol{r}',\boldsymbol{r})\,r^{2p}e^{-r^2/2}Y_{00}(\Omega)$$

$$a_{LO} \leftrightarrow (p',p) = (0,0) \qquad a_{NLO} \leftrightarrow (p',p) = (0,1)\ \text{or}\ (1,0) \qquad \text{etc.}$$

- Operators are defined in P and generated by a progression in oscillator quanta

$$V_\delta^S = a_{LO}^{\rm S}\delta(\boldsymbol{r}) + a_{NLO}^{\rm S}\left(\boldsymbol{a}^\dagger \odot \boldsymbol{a}^\dagger\ \delta(\boldsymbol{r}) + \delta(\boldsymbol{r})\ \tilde{\boldsymbol{a}} \odot \tilde{\boldsymbol{a}}\right) + \cdots$$

(If you are not comfortable with this, but the dimensions back in - all coordinates and derivatives are scaled by b - then take $b \to \infty$ (recover momentum basis physics)

$$\to a_{NLO}^{S}(\overleftarrow{\nabla}^2\delta(\mathbf{r}) + \delta(\mathbf{r})\overrightarrow{\nabla}^2)$$

The interaction H^{IR}

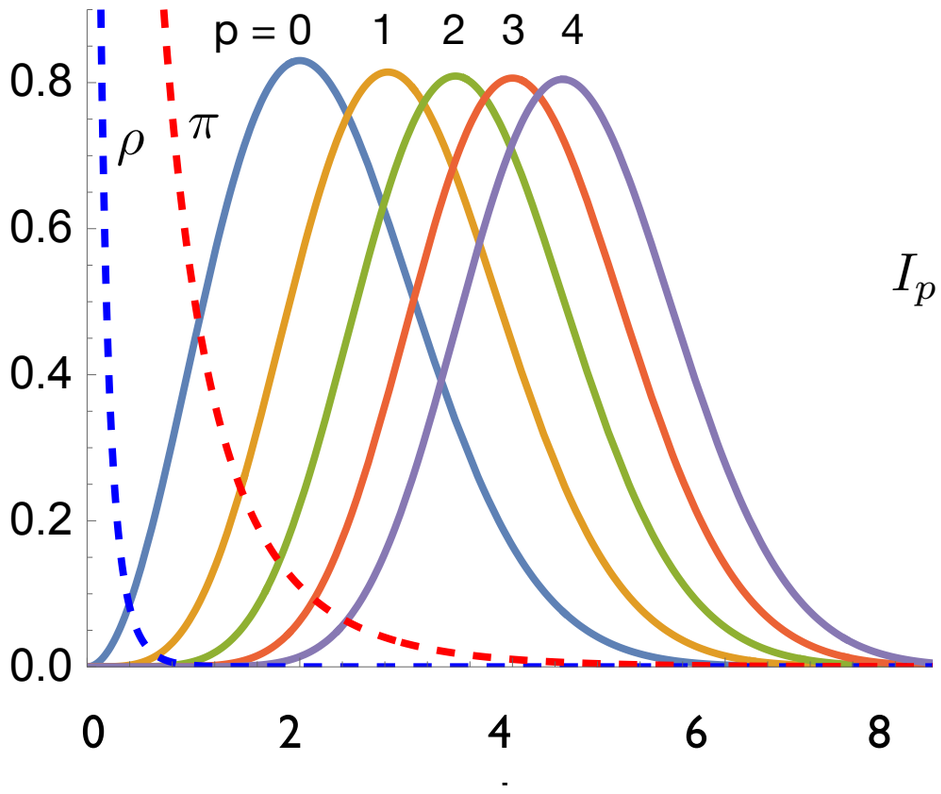
- This interaction comes from the KE and the IR contribution of V_π
- Several equivalent expressions for V_π^{IR}

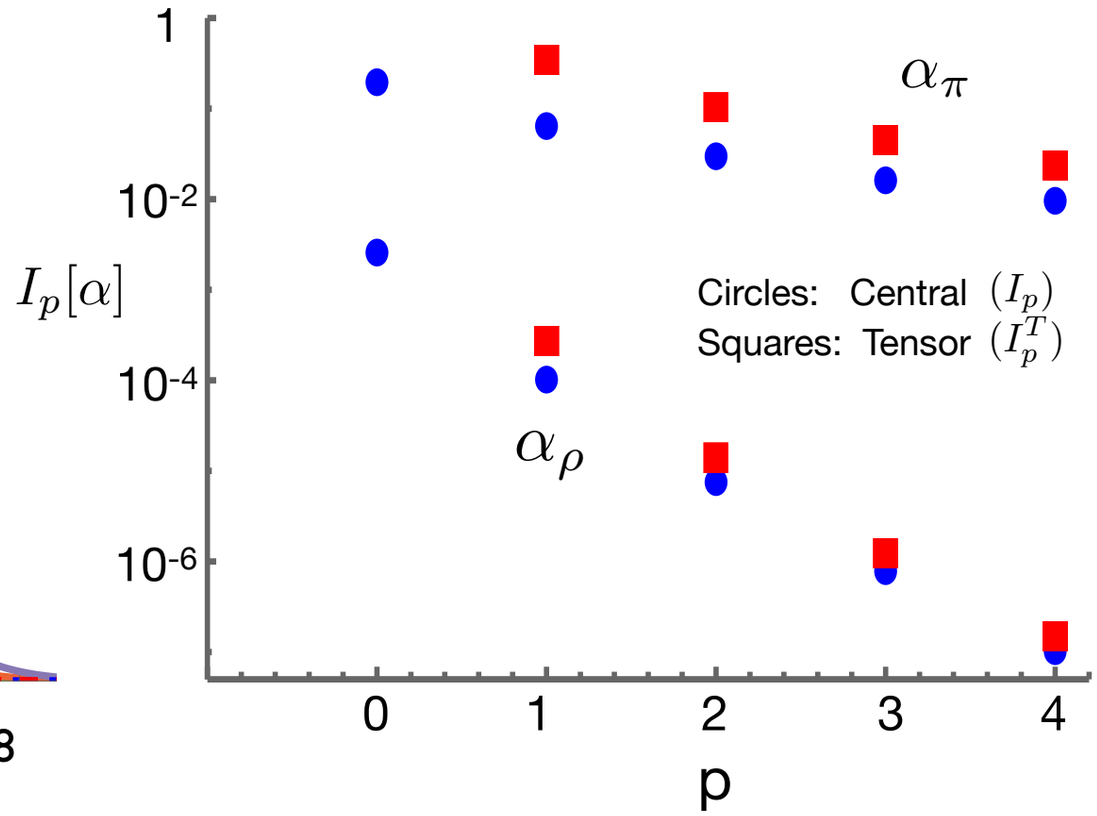
$$V_\pi^{IR} = \sum_{\alpha\beta > N^3LO} |\alpha >< \alpha| V_\pi^{subtracted} |\beta >< \beta|$$

in the Talmi integral expansion of this matrix element, one retains ONLY contributions beyond the order of the short-range expansion

- Supplies *all Talmi integrals not supplied* by the UV operator expansion
- Intuitively as one increases the fidelity of the short range expansion, the missing physics one must provide should *diminsh:* HOBET behaves this way

- We work at N3LO, so V_π^{IR} is zero until we reach $p \geq 4$





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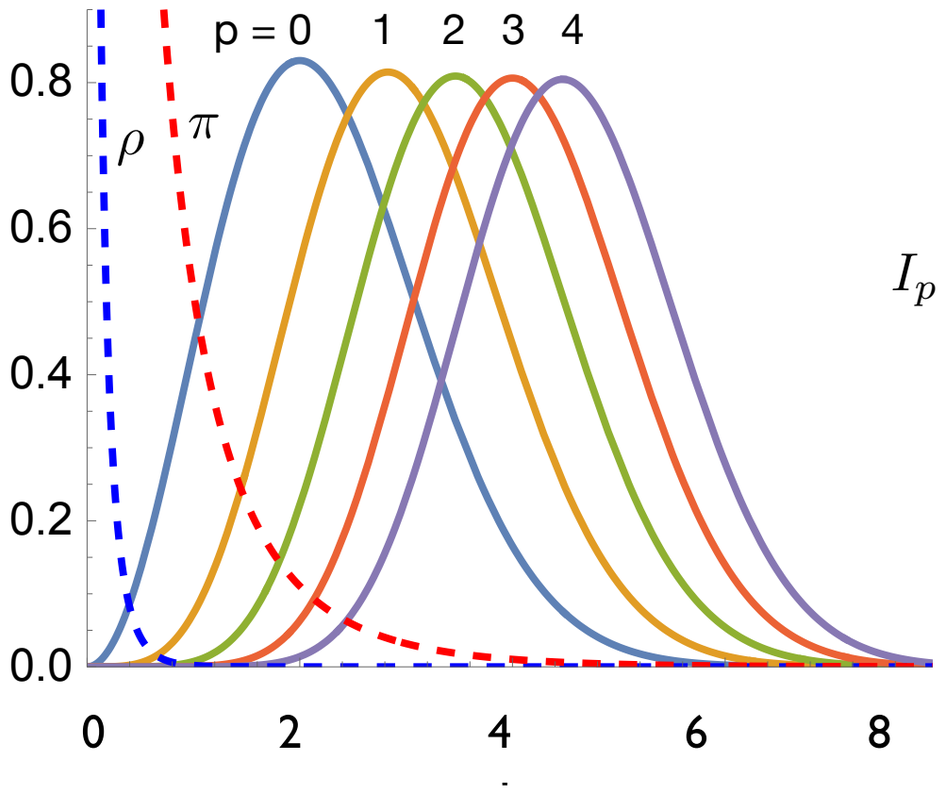
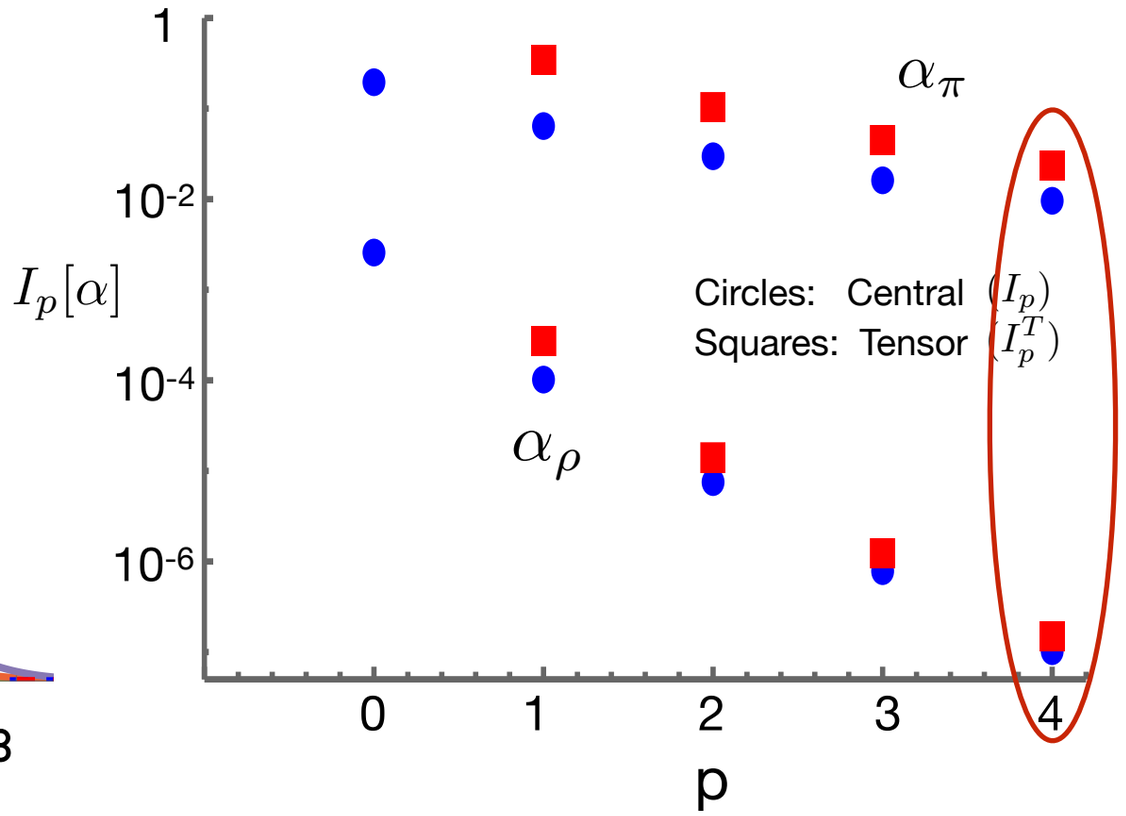
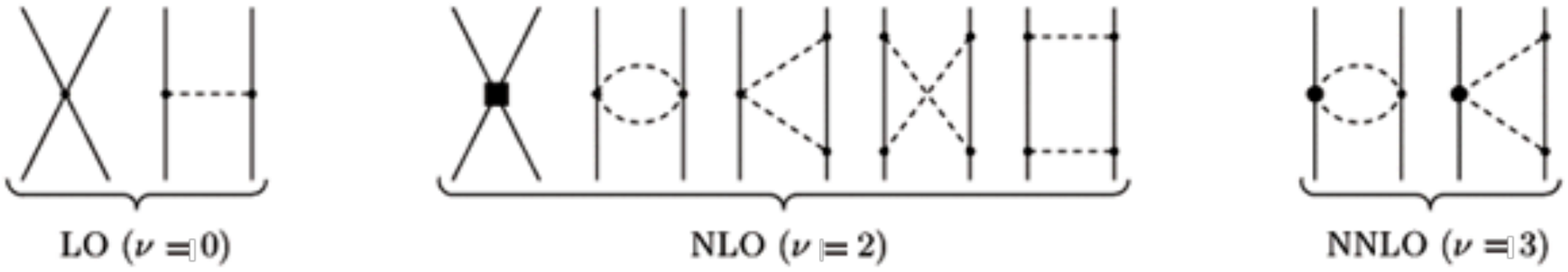




- One is free to calculate this



- But numerically the short-range fidelity achieved at N3LO allows

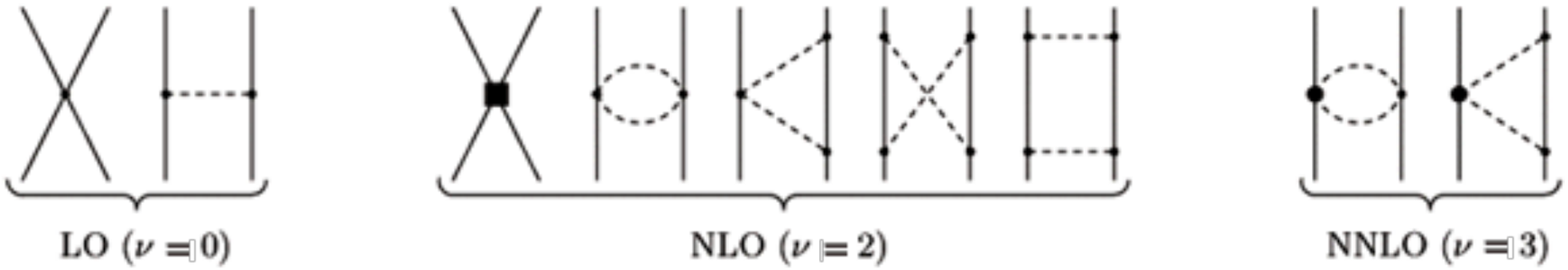
$$V_\pi^{subtracted} \to V_\pi^{OPEP\ subtracted}$$ (we verify empirically)

- Power counting, central and tensor interactions (illustrated for a potential)

$$a_{order}^{central} \propto I_{order} \qquad a_{order}^{tensor} \propto (1 - \frac{6}{\alpha^2}) I_{order} + \frac{6}{\alpha^2} I_{order-1} \qquad \alpha \equiv \sqrt{2} m b = \begin{cases} 9.4 & m_\rho \\ 1.7 & m_\pi \end{cases}$$

- Central interactions and vector-meson-range tensor interactions have a simple power counting

- One is free to calculate this



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- But the tensor pion channel breaks the pattern
  - Of no consequence for a pion-full theory, as all pion LECs are added
  - But a pionless theory will converge in triplet channels more slowly, without the addition of one pion LEC needed to restore the counting

Evaluation: H^{IR}

- The IR interaction is known, and relatively easy to evaluate, as the matrix elements of the free KE can be evaluated analytically

$$H^{IR} = T_{12} + V_\pi^{IR}$$

$$H^{eff\ IR} = H^{IR}\frac{E}{E - QH^{IR}} = T\frac{E}{E - QT} + \frac{E}{E - TQ}V_\pi^{IR}\frac{E}{E - QT} + \ldots$$

$$\frac{E}{E - QT} = \frac{E}{E - T}\left[P\frac{E}{E - T}P\right]^{-1}$$

so that the numerical cost of evaluation the Green's function is a matrix inversion in P: HO matrix elements of the free G are analytic

- Thus the KE is summed to all orders, which it must be to correctly capture the physics as E $\rightarrow$ 0

Continuum states: How is H^{eff} (the LECs) determined?

- E>0 so there are two scattering solutions that behave well at infinity
- We pick an energy E: a HOBET solution at that energy is guaranteed (except for a set of discrete energies known as the Kohn singularities)
- We look up the phase shift, and build the associated Green's function: we have fixed the correct IR behavior
- We solve the BH equation for this E and for some starting H^{eff}, e.g., LO
- We get an eigenvalue other than E. What can be wrong? It is not the IR, but the UV content of H^{eff}
- We adjust a_{LO} until the eigenvalue becomes E

The algorithm implemented selects the relevant phase shift information

- In our procedure we insert in our error budget a "cost function" which uses naturalness to estimate the contribution of all omitted orders

- In the first iteration we pick a low-energy point, determine a_{LO} , then step gradually to higher energies until we see a discrepancy in the energy self-consistency that approaches the cost function

- We take that second point, and use two points to determine a_{LO}, a_{NLO}

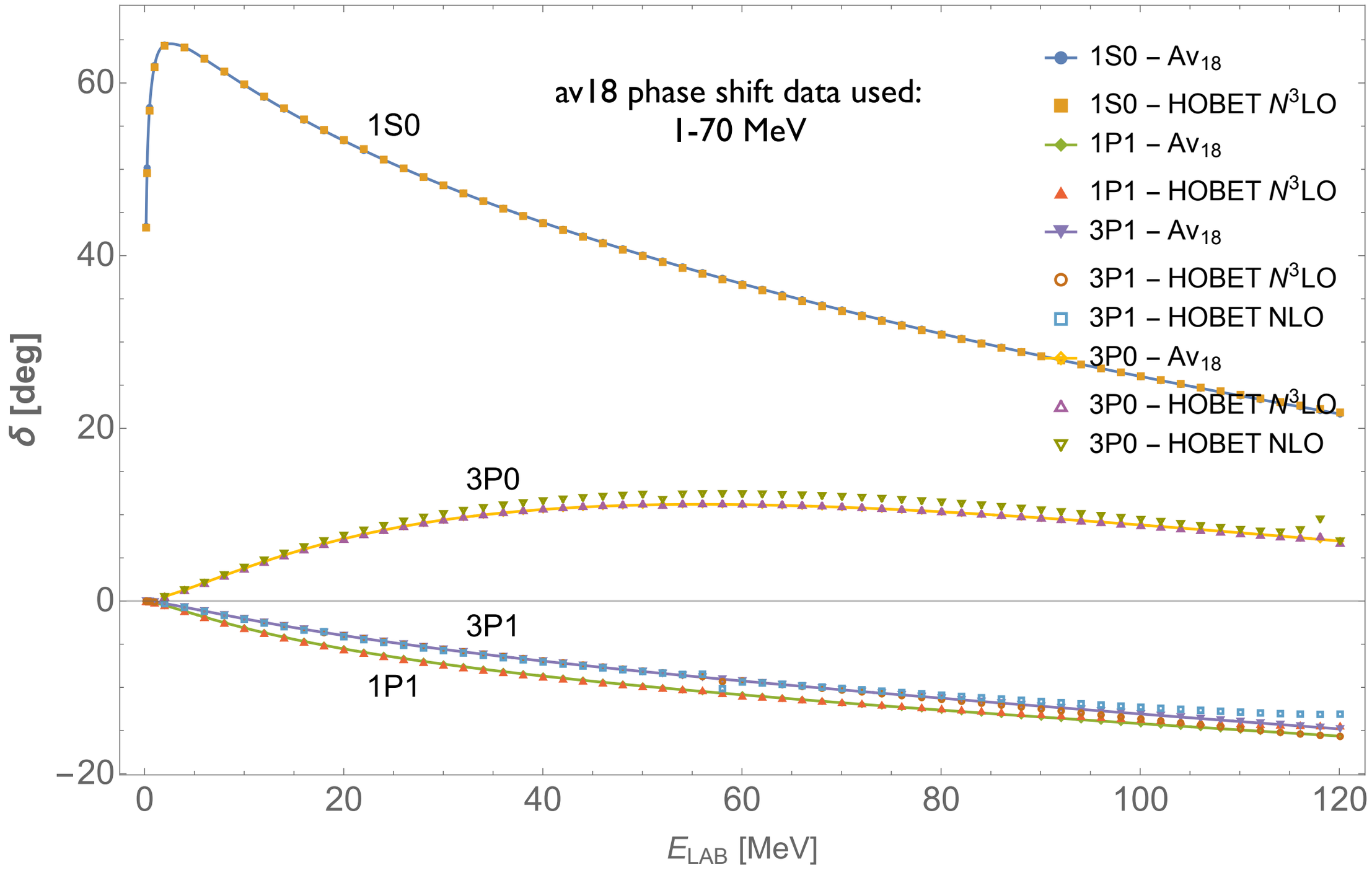
- Then repeat: when fitting a new LEC, we find other LECs remain stable except for the LEC last fit — this is the proper behavior of a good theory

- The procedure tells one what information is relevant: data to $\sim$ 70 MeV lab

- With H^{eff} now fixed, we can predict bound-state eigenvalues

How do bound states work?

- HOBET in analytic in E, so the energy-independent continuum LECs are the bound LECs
- The Green's function is simple: only one solution falls off at infinity
- The energy in the Green's function must be the eigenvalue
- Make an initial guess at the energy, generate the eigenvalue, then loop, using the eigenvalue as the new energy
- Convergence to self-consistency is rapid, typically to machine precision in 4-6 iterations
- Solutions exist at discrete energies
- The N3LO continuum calculation yields a deuteron binding energy of -2.2278 MeV. The N3LO error is 3 keV or about 0.1%


av18 phase shift data used:
1-70 MeV
1S0
3P0
3P1
1P1
1S0 – Av18
1S0 – HOBET N3LO
1P1 – Av18
1P1 – HOBET N3LO
3P1 – Av18
3P1 – HOBET N3LO
3P1 – HOBET NLO
3P0 – Av18
3P0 – HOBET N3LO
3P0 – HOBET NLO
δ [deg]
60
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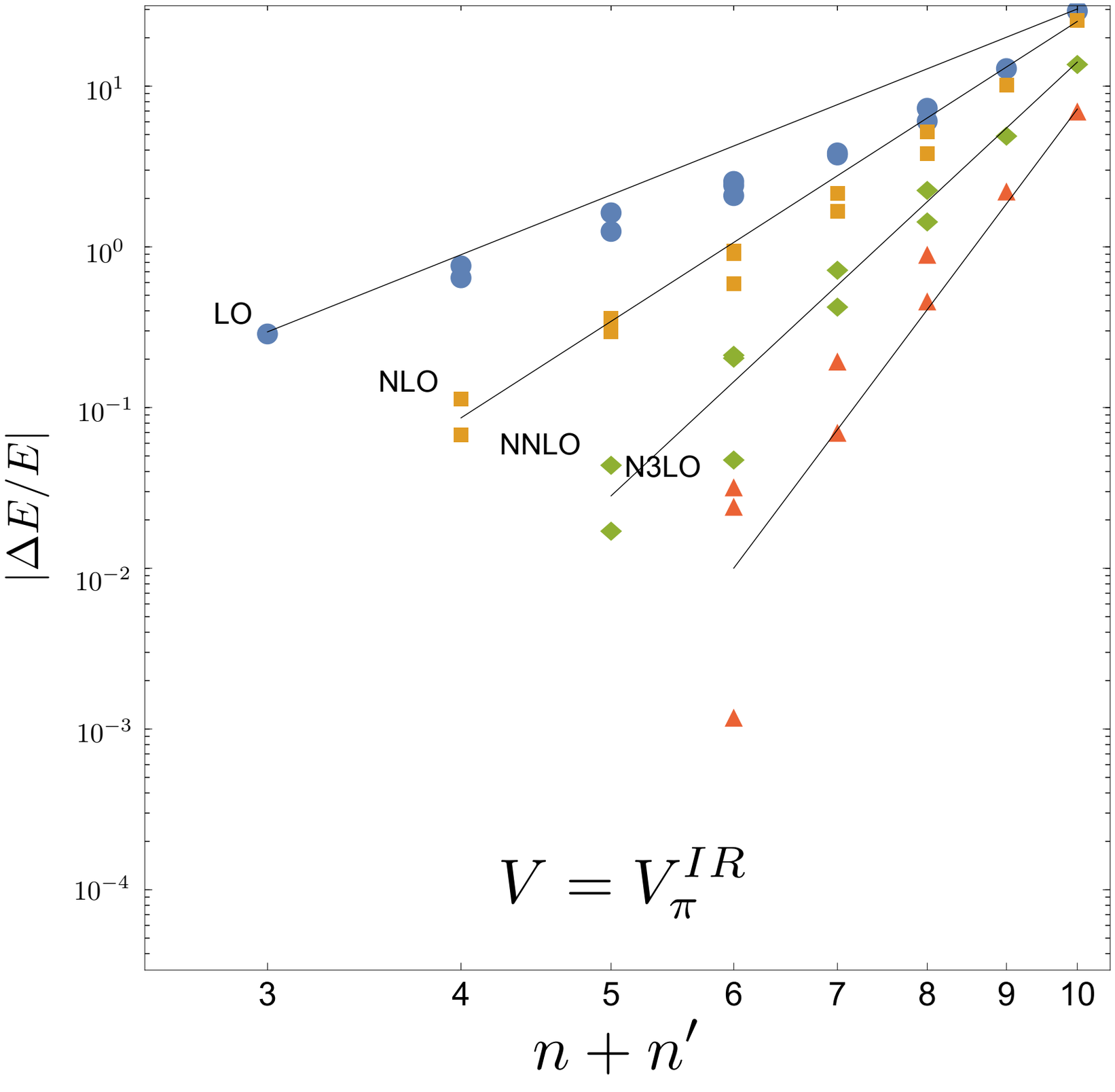
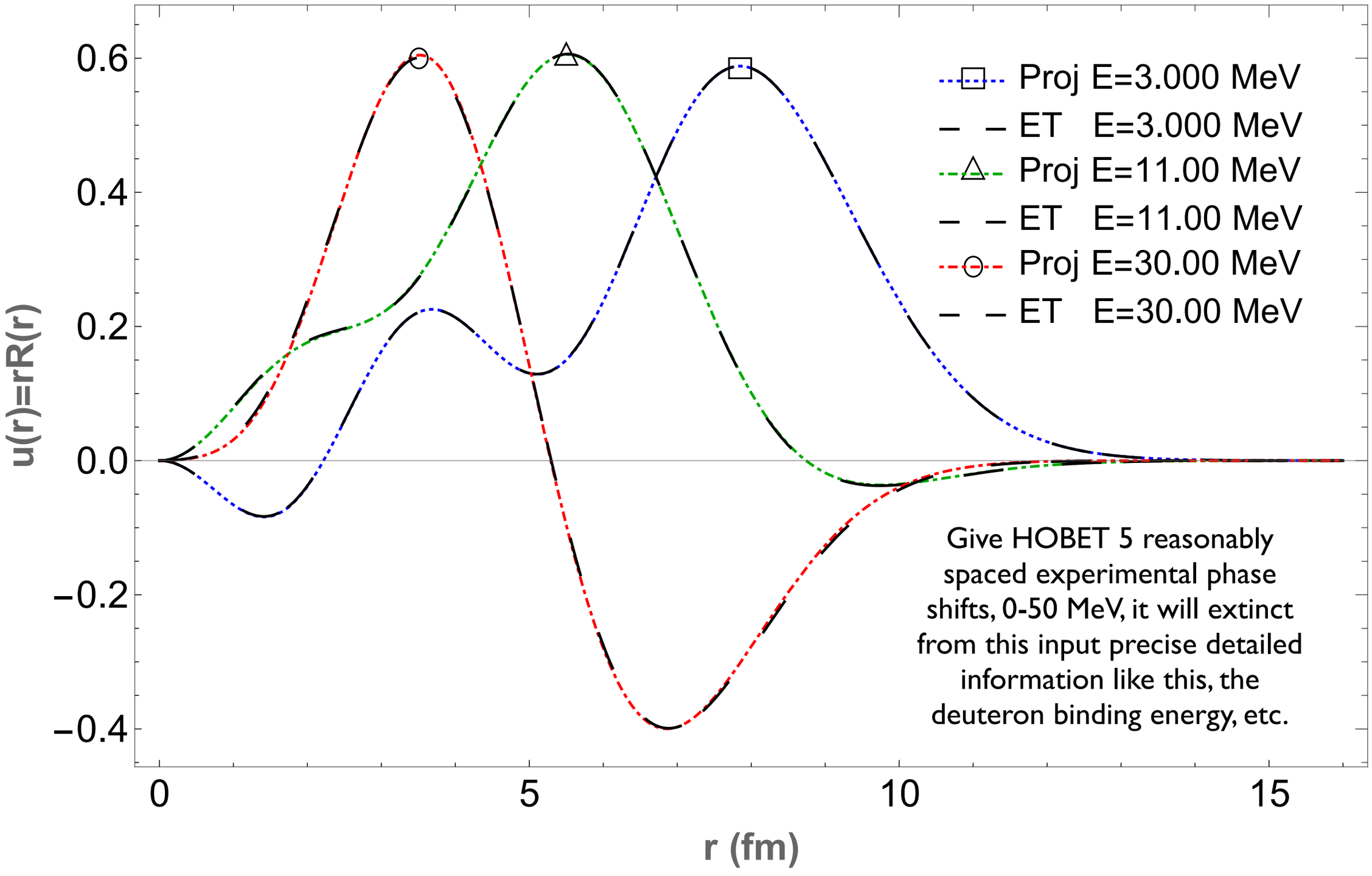
0
−20
0
20
40
60
80
100
120
ELAB [MeV]



Test using av18 phase shifts, then comparing to Heff matrix elements calculated numerical from av18



The calculated projected (onto P) s-wave scattering wave functions evaluated for energies not used in the fitting, compared to exactly calculated av18 solutions:
the results are not distinguishable

Nuclear Calculations

- HOBET was influenced by a paper that was impactful at its time

  Gomes, Walecka, Weiskoff, "Properties of Nuclear Matter", Annal Phys 3 (1950)

- Sought to explain how hard-core scattering of nucleons could be compatible with the success of mean-field descriptions of nuclei or nuclear matter

- Picture emerged that nucleon pairs spend a short time at short distances, scattering repeatedly, then "heal" to a soft state through IR interactions

- This is basically the picture that emerges when HOBET is extended beyond A=2: it guides re-summations that connect nuclear systems to A=2

- HOBET allows one to subtract from the A-body problem all of the physics constrained by A=2, which includes most of the 3-body physics: only a suppressed 3-body contribution remains

Outline of Procedure

- For illustration, consider A=3

- If our LECs were derived from A=2, with $\Lambda_1 + \Lambda_2 \leq \Lambda$, then to avoid the need for information that has been integrated out

$$\Lambda_1 + \Lambda_2 + \Lambda_3 \leq \Lambda$$

- The reorganization is similar to the following (two-body contribution to H^{eff})

$$H_{12}^{A=3} = P_3 \left[H_{12} + H_{12} \frac{Q_3}{E} H_{12} + H_{12} \frac{Q_3}{E} H_{12} \frac{Q_3}{E} H_{12} + ... \right] P_3$$

$$= P_2 \left[H_{12} + H_{12} \frac{Q_3(P_2+Q_2)}{E} H_{12} + H_{12} \frac{Q_3(P_2+Q_2)}{E} H_{12} \frac{Q_3(P_2+Q_2)}{E} H_{12} + ... \right] P_3$$

- Resume according to the number of P_2 insertions

- Each such term can be put into WH-Luu form: IR propagation, hard scattering, IR propagation: one repeatedly returns to P_2 while staying in Q_3

- But P_2Q_3 depends on Λ_3 : $\Lambda_1 + \Lambda_2 + \Lambda_3 > \Lambda$ + $\Lambda_1 + \Lambda_2 \leq \Lambda$

- The two-body contribution is entirely determined by two-body LECs, and takes the form

$$H_{12}^{eff} = \frac{E}{E - TQ} T + V_{12}^{eff}(\Lambda_3)$$

- But one does not see this in the literature, though its form is required

- The same procedure can be employed for the rest of the interaction, the Faddeev series (*all of which is three-body from the perspective of Q_3*)
  - Much of this can be rewritten in terms of H_{12}^{eff}
  - A three-body remainder that cannot be related to two-body physics

- We have carried this out for A=3,4. Neglecting the three-body term

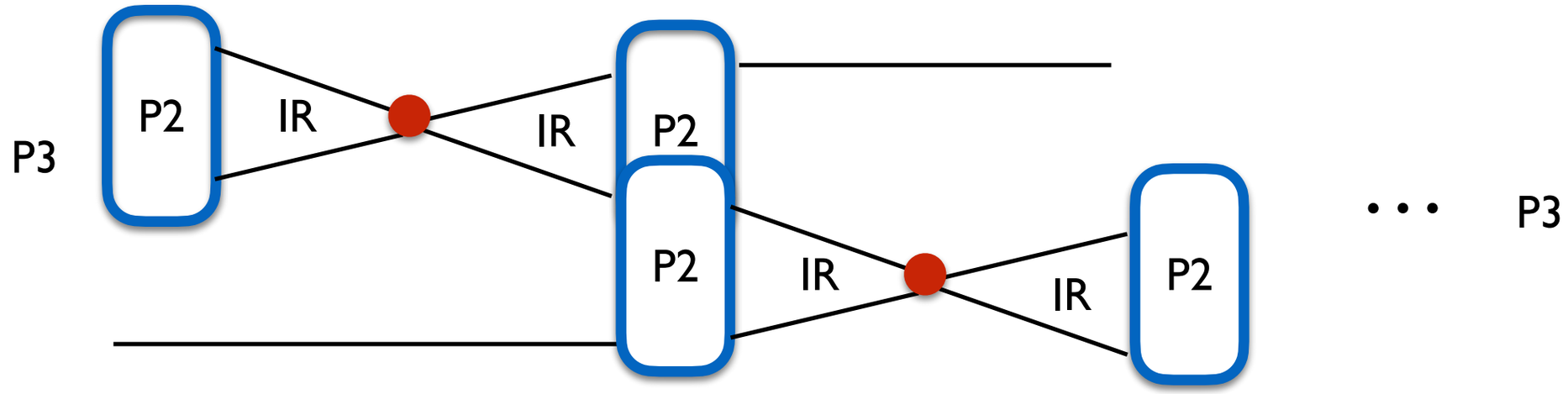
$$^3\mathrm{He}: \quad \mathrm{HOBET}: \quad -6.884\ \mathrm{MeV} \qquad av18\ \mathrm{exact}: \quad -6.928\ \mathrm{MeV}$$

$$^4\mathrm{He}: \quad \mathrm{HOBET}: \quad -24.103\ \mathrm{MeV} \qquad av18\ \mathrm{exact}: \quad -24.22\ \mathrm{MeV}$$

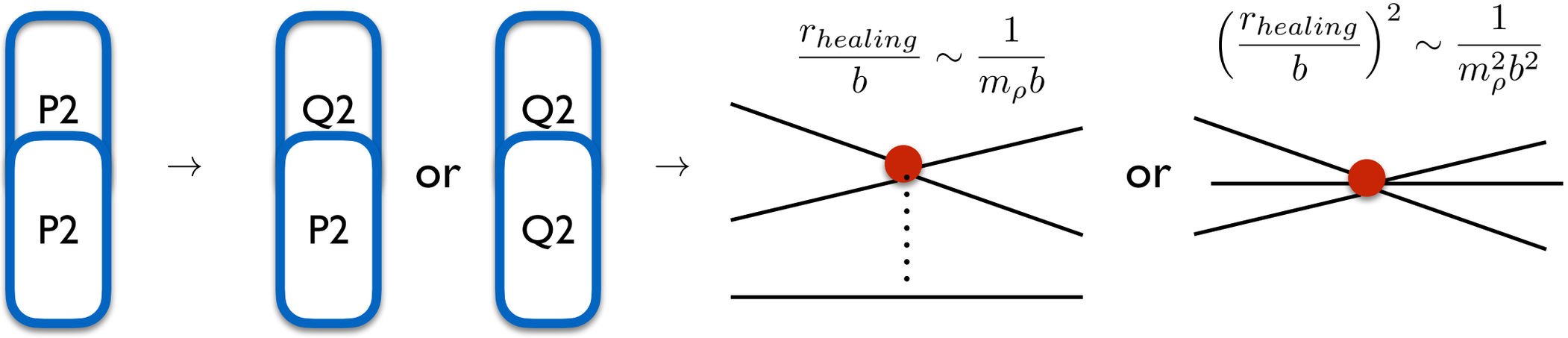
McElvain and WH (preliminary)

The difference is small compared to the "true" three-body physics, e.g., Urbana IX

- The three-body physics summed is of the form, where all intermediate states $\in Q_3$



and terms where the nucleon pair after scattering does not "heal", e.g., where the pair remains in a high momentum state until interacting again with another nucleon, cannot be determined from our two-body results, and would need to be treated as a 3-body short-range operator. Those interactions involve the replacements



so all of this is familiar, except HOBET would never worry about C_D or C_E, but only about the iteration to all orders, as this is what connects to observables

Summary

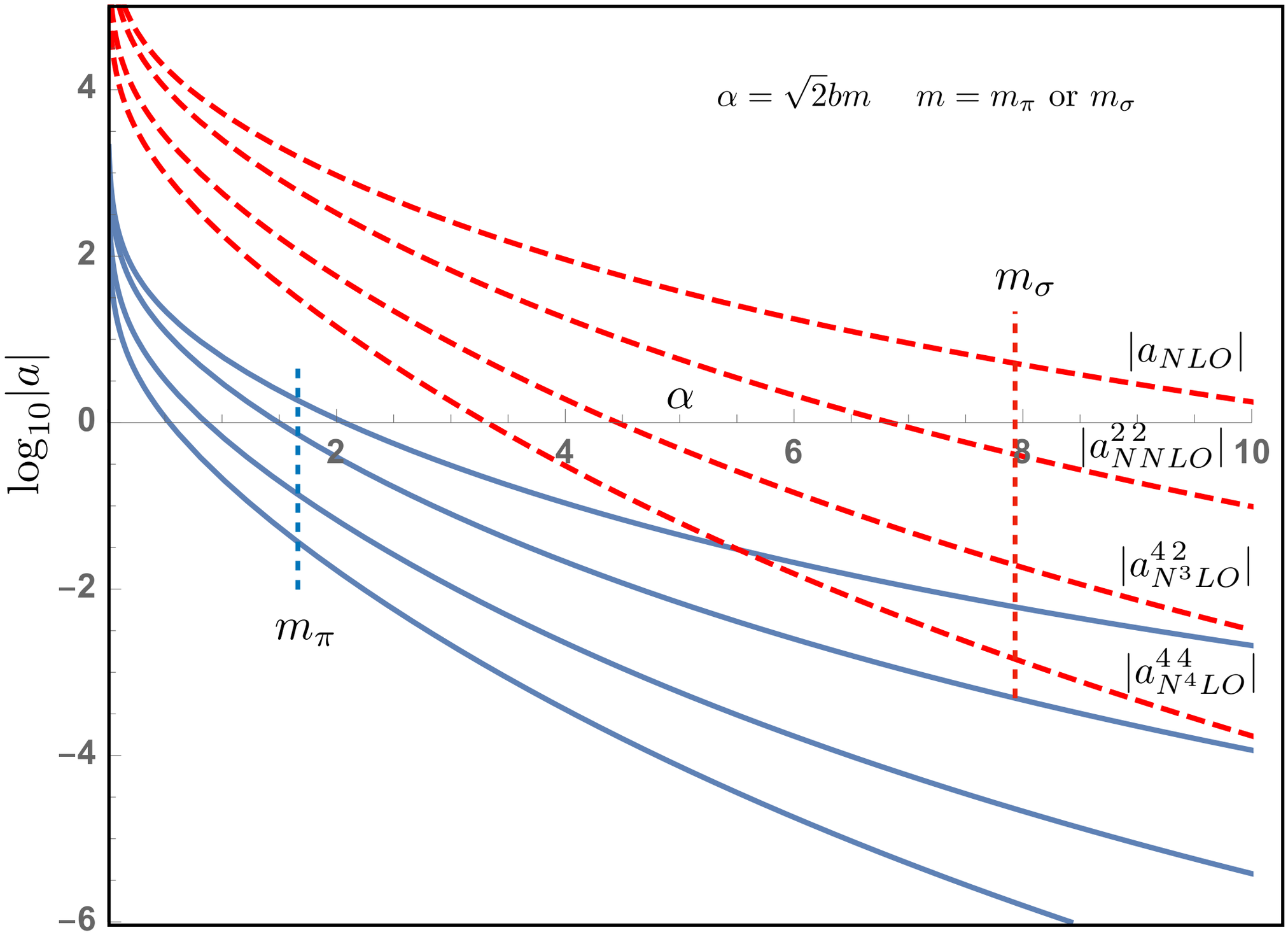
- The field has most of its investment in one approach to EFT/nuclear physics, where the challenges include
  - the large gap between the chiral potential and nuclear scales
  - the cutoff dependence
  - the nonintuitive behavior of the pion physics
  - the use of a plane-wave basis where technically all interactions are short-range

- HOBET has had little investment - 3 students - but its attributes are attractive
  - a direct reduction of QCD to the nuclear scale/basis
  - cutoff independence
  - translational invariance
  - energy continuity: equivalent treatment of bound states and reactions
  - an elegant procedure for fixing LECs accurately from the minimum possible data
  - the absence of intruder states, opening up perturbative possibilities
  - good numerical results in the limited applications done to date

As in the stock market, diversification is usually beneficial — even though each approach has the overhead of having to invent new techniques

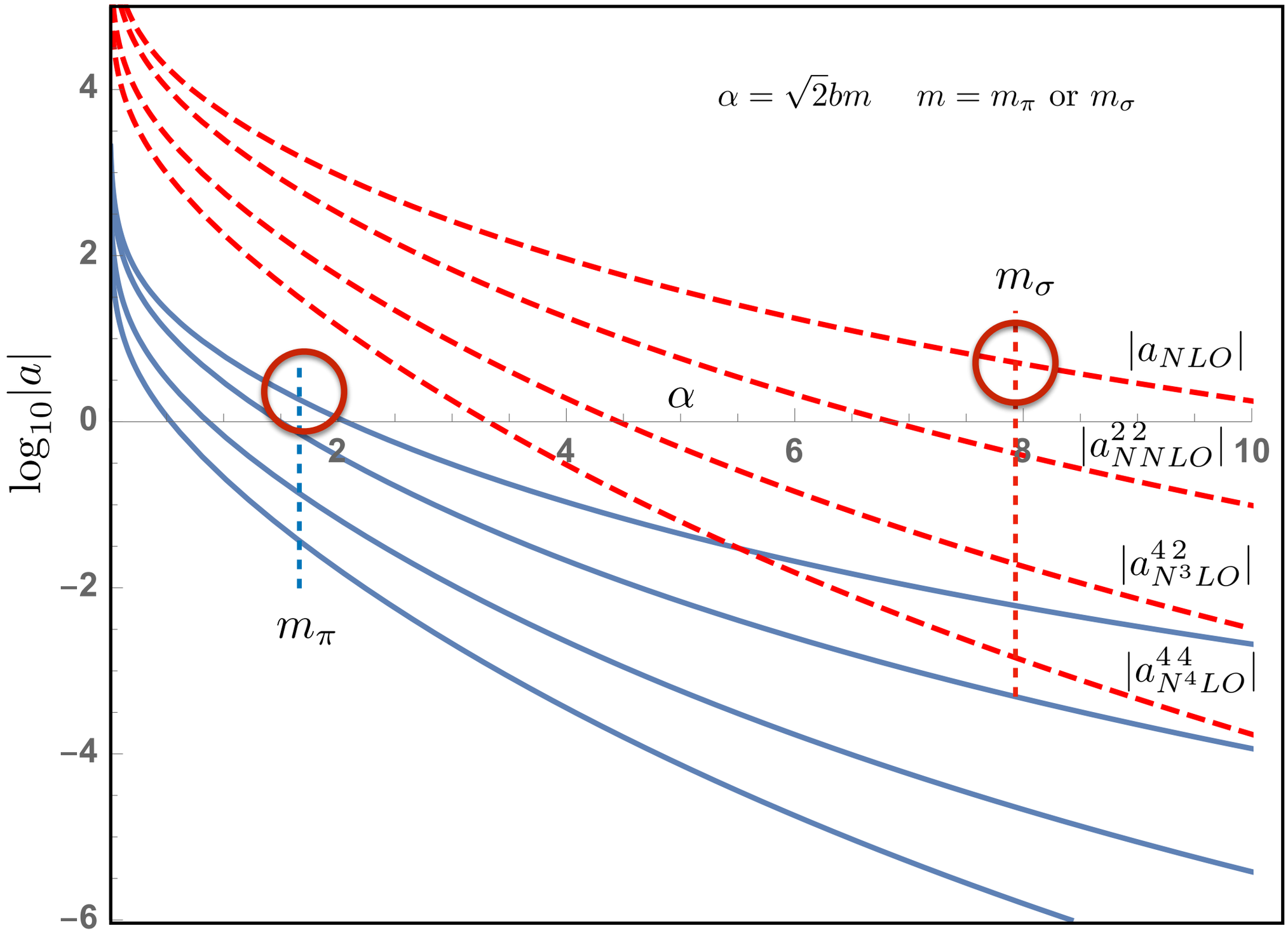
Backup Slides

One pion vs two pion exchange



One pion vs two pion exchange

LO



One pion vs two pion exchange

N^3LO