Classical and Quantum Computing of Shear Viscosity

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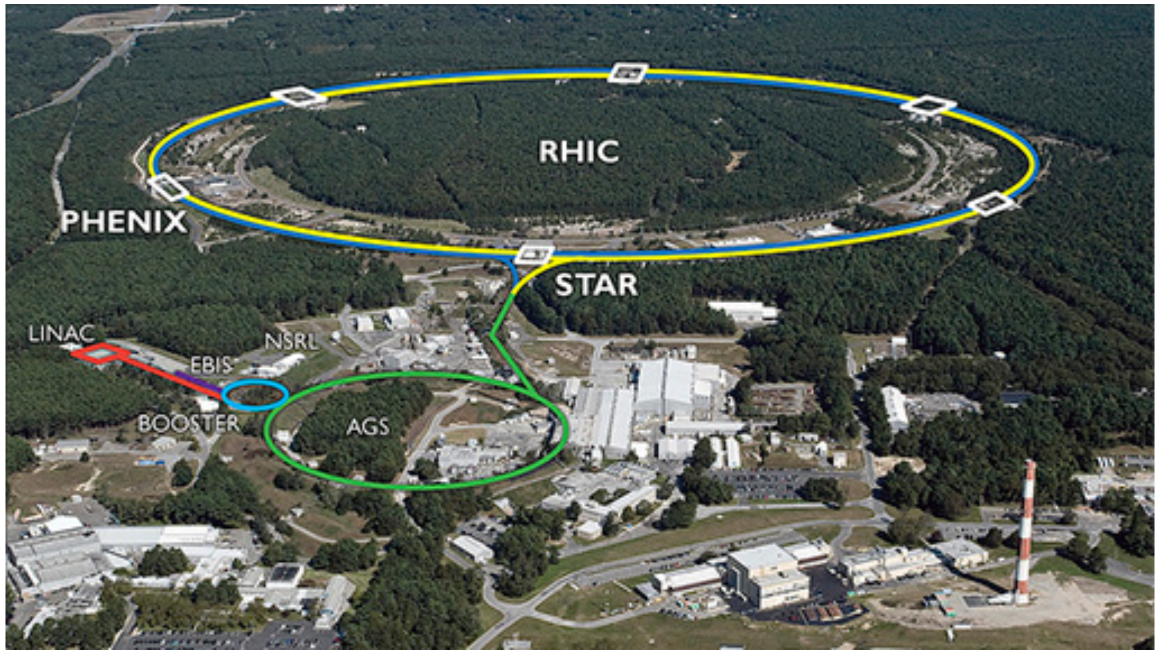
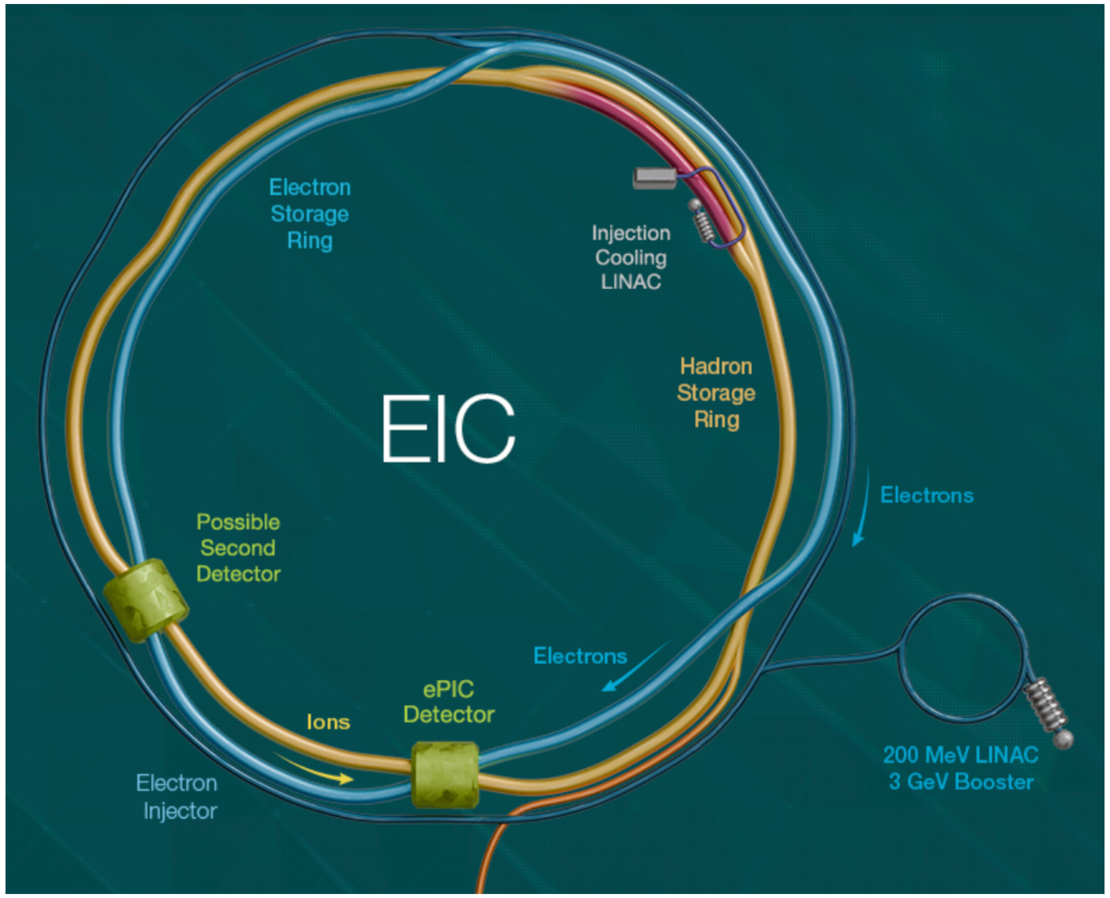
Quantum Computing for High Energy QCD



Relativistic Heavy Ion Collider (RHIC)
at Brookhaven National Laboratory



Electron-Ion Collider (EIC) built upon RHIC

- **High energy collider physics of QCD: mixture of perturbative & nonperturbative**

Perturbative scales: jet energy, heavy quark mass, high temperature, gluon saturation

Nonperturbative inputs: PDF, TMD, fragmentation, **transport coefficient**, equation of state

- **Quantum computing useful for nonperturbative calculations where Euclidean lattice QCD suffers from sign problem: real time and high density**

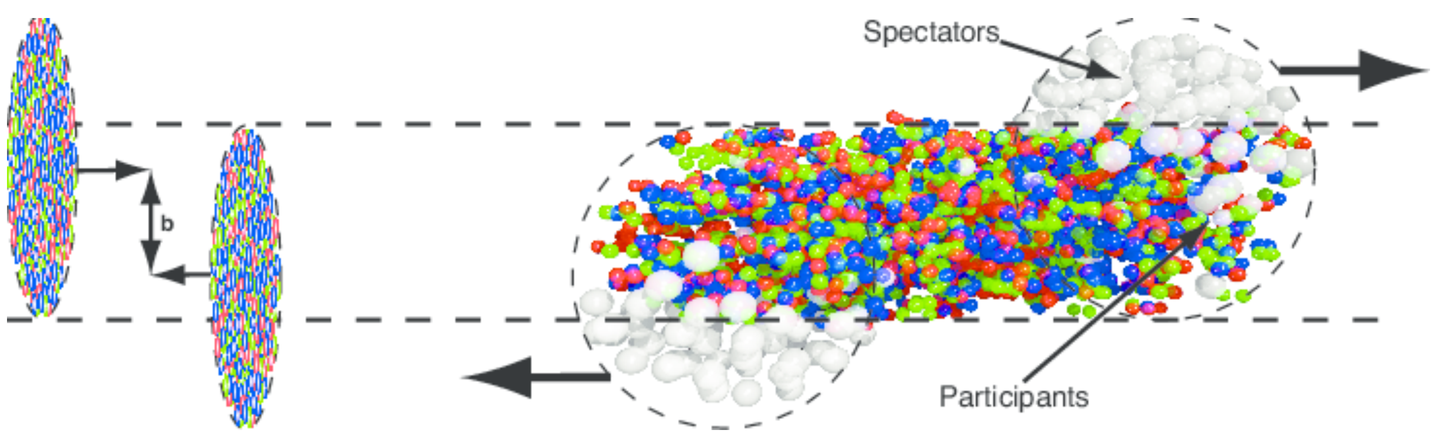
Contents

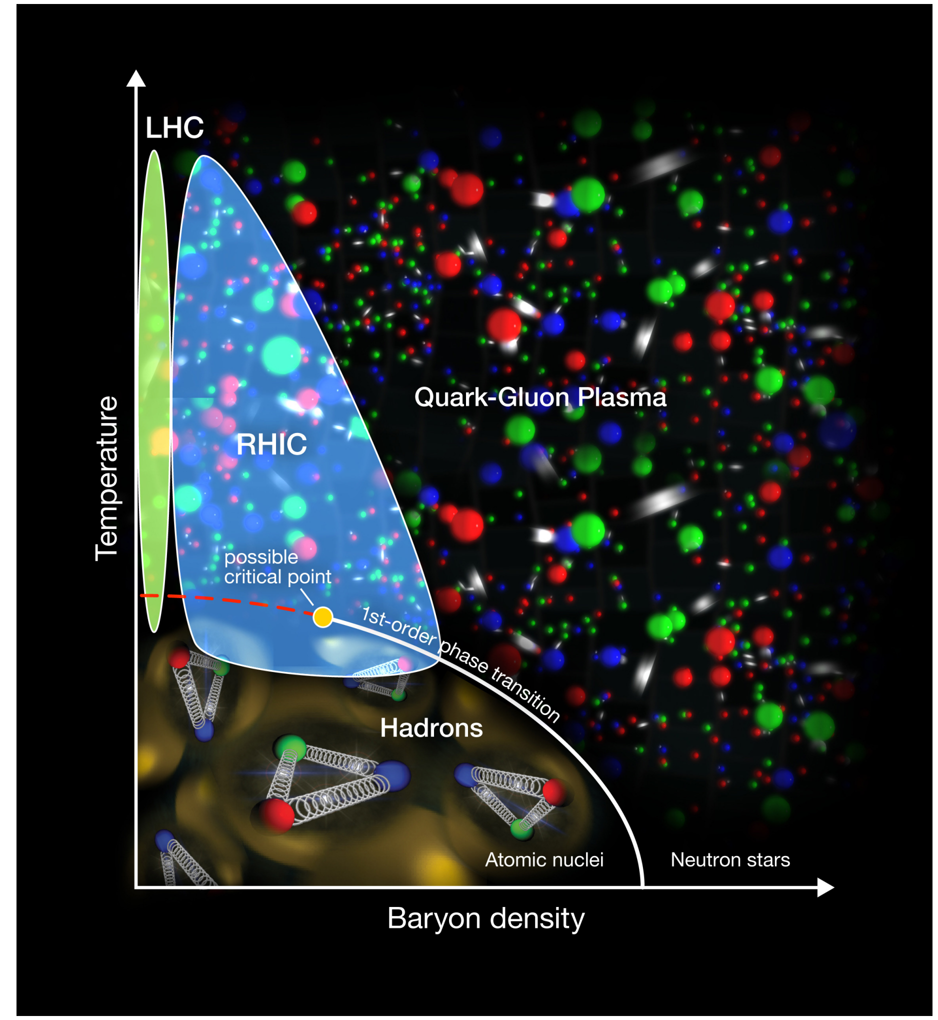
- Calculating shear viscosity from **real-time Hamiltonian lattice** approach
  - Motivation
  - Calculation setup
  - Quantum algorithm
  - Example: 2+1D SU(2) pure gauge theory
  - Classical and quantum computing results on a small lattice

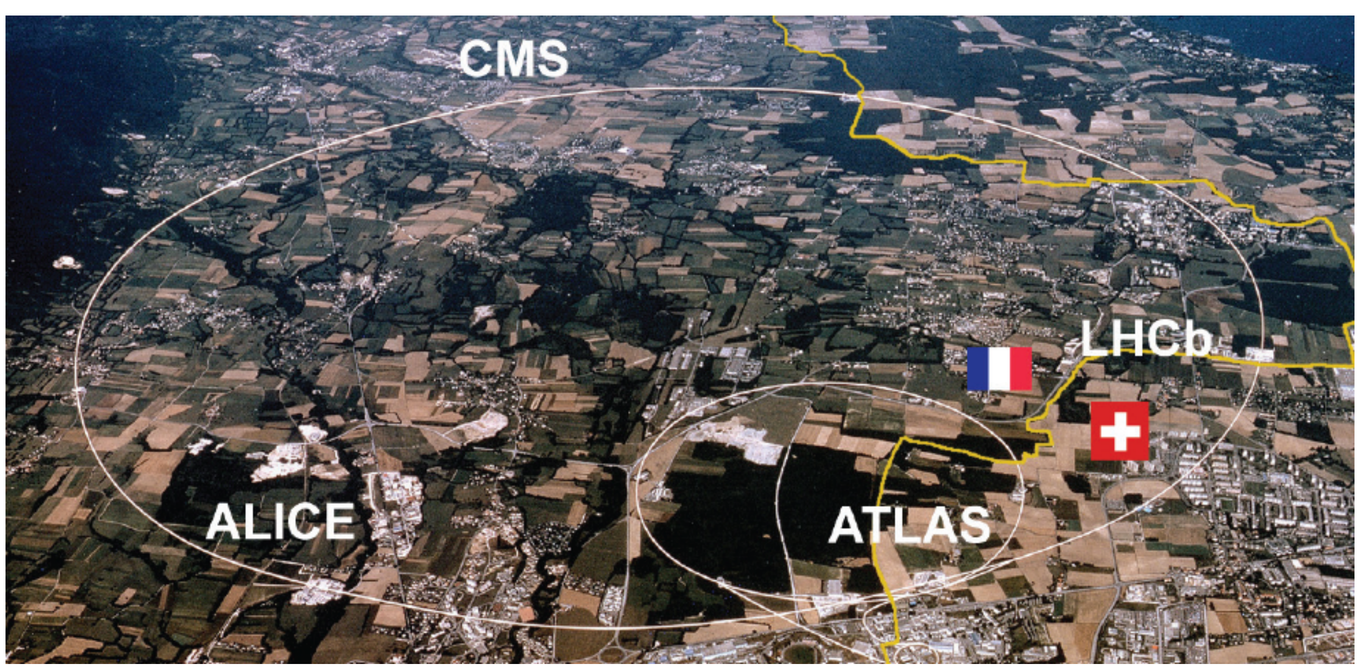
Motivation

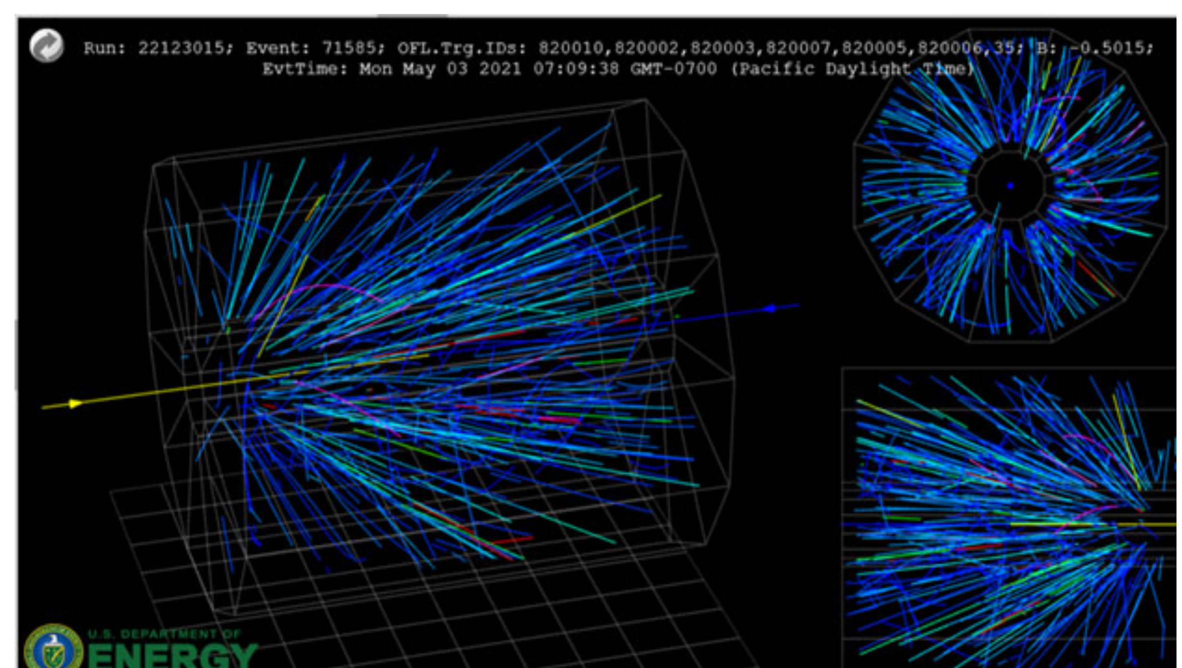
Introduction of Heavy Ion Collisions

- **Relativistic heavy ion collisions: study deconfined phase of nuclear matter governed by strong interaction (QCD): quark-gluon plasma (QGP), $T > 150$ MeV**

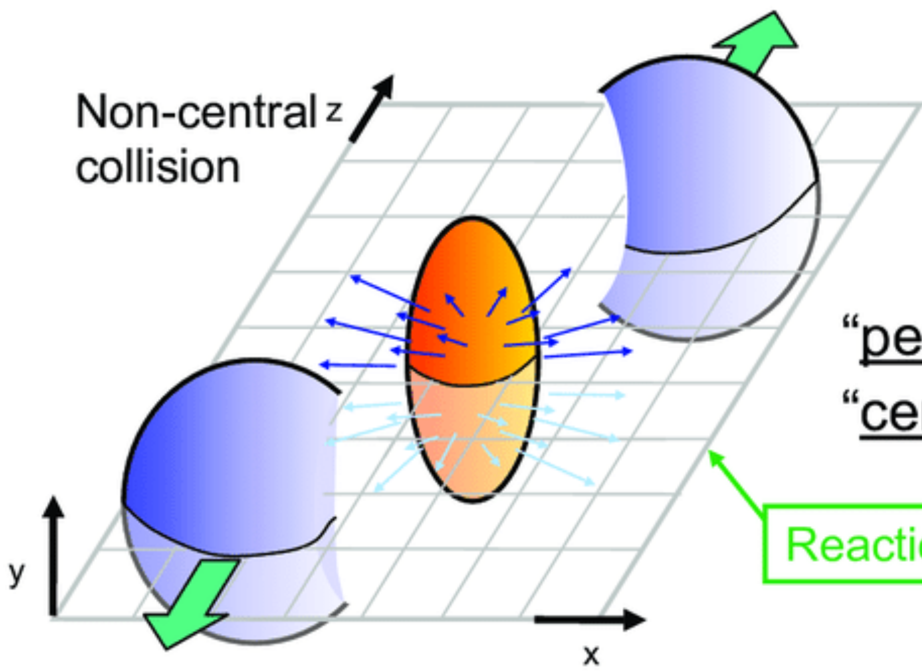




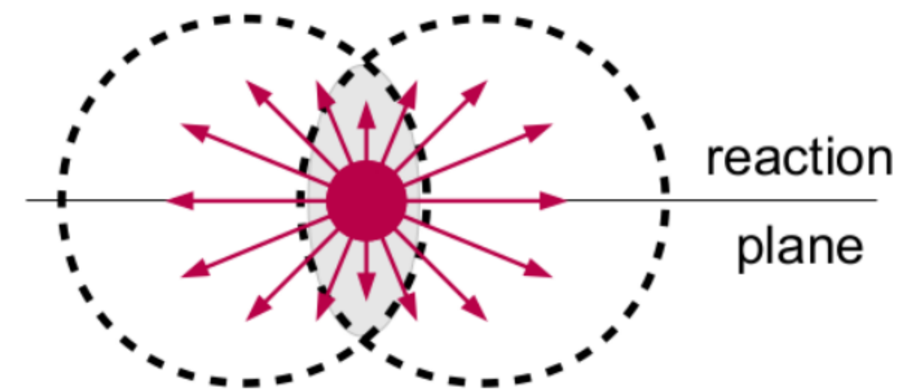




Particle Distribution in Azimuthal Plane



"peripheral" collision (b ~ b_{max})
"central" collision (b ~ 0)

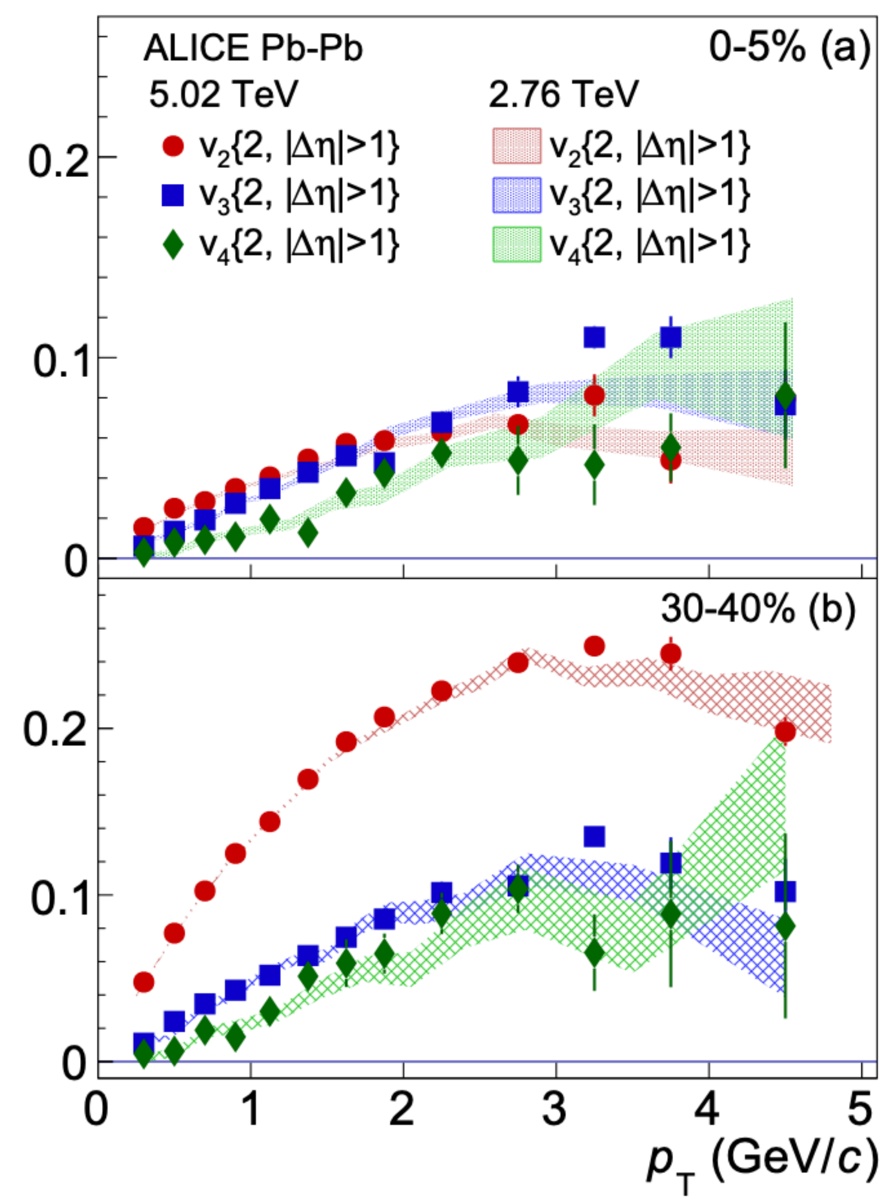


- **Anisotropic distribution —> collective flow**

$$\rho(\phi) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} v_n \cos(n\phi)\right]$$

Flow coefficients

v_2: elliptic flow,
v_3: triangular flow



ALICE 1602.01119

Hydrodynamics and Shear Viscosity

- **Use relativistic hydrodynamics to describe collective behavior**

$$\nabla_\mu T^{\mu\nu} = 0$$

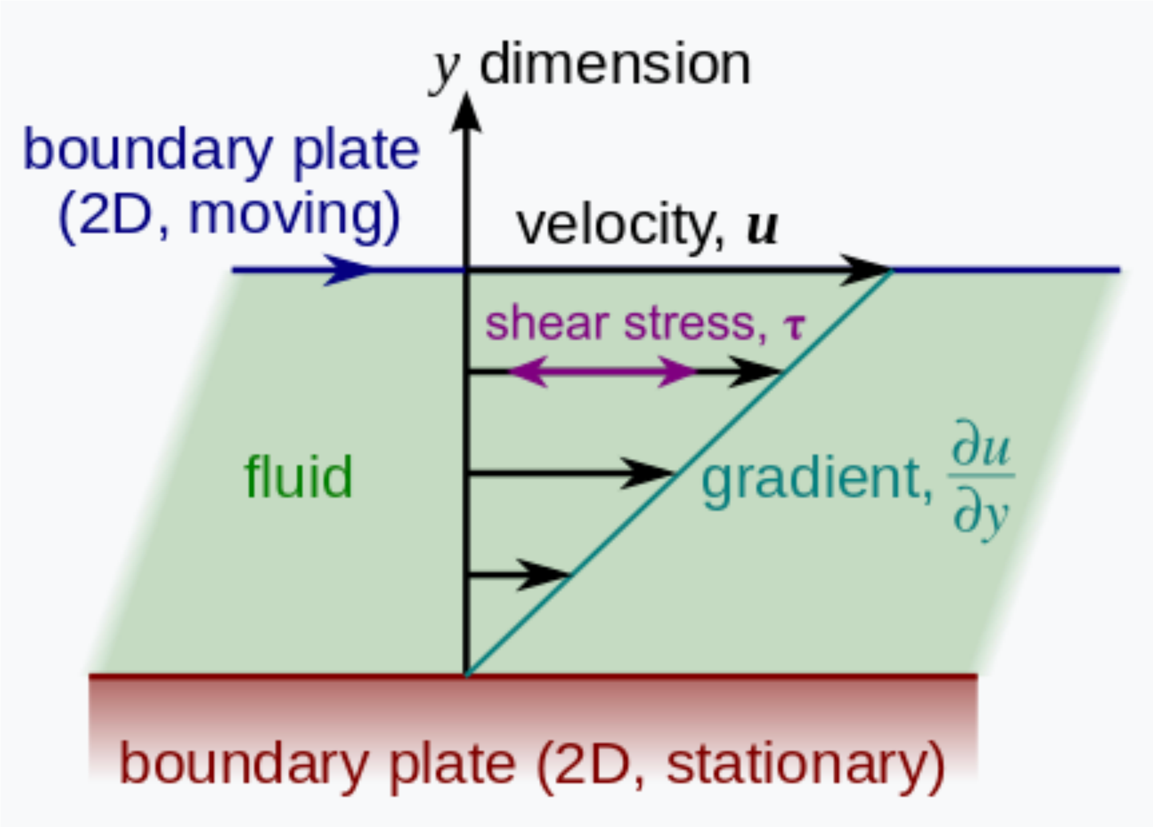
$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu} + 2\eta \nabla^{\langle\mu} u^{\nu\rangle}$$

$$2\nabla^{\langle\mu} u^{\nu\rangle} = \Delta^{\mu\rho}\nabla_\rho u^\nu + \Delta^{\nu\rho}\nabla_\rho u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\rho u^\rho \qquad \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$$

Make it causal: Israel-Stewart hydrodynamics

- **Shear stress and viscosity η**

$$F = \eta A \frac{\partial u}{\partial y}$$



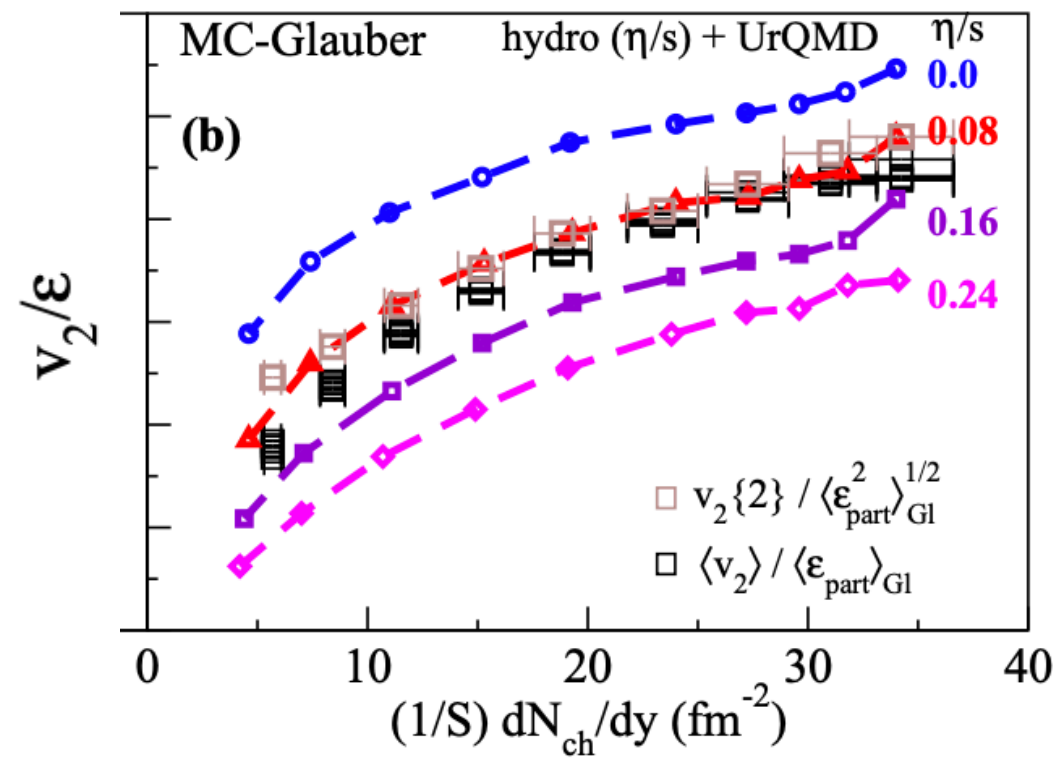
Anisotropic Flow and Shear Viscosity

- **Hydrodynamic calculations indicate QGP has small shear viscosity**

$\eta/s = 0.08$ best describes data

$\eta/s \sim 1000$ for air

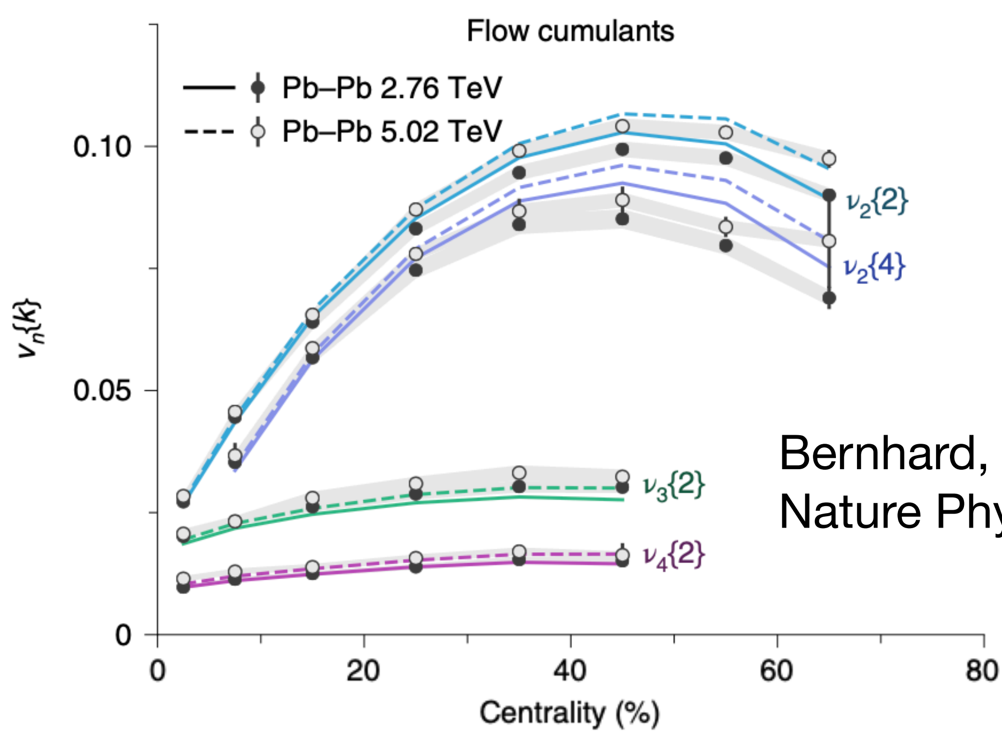
$\eta/s \sim 10$ for water



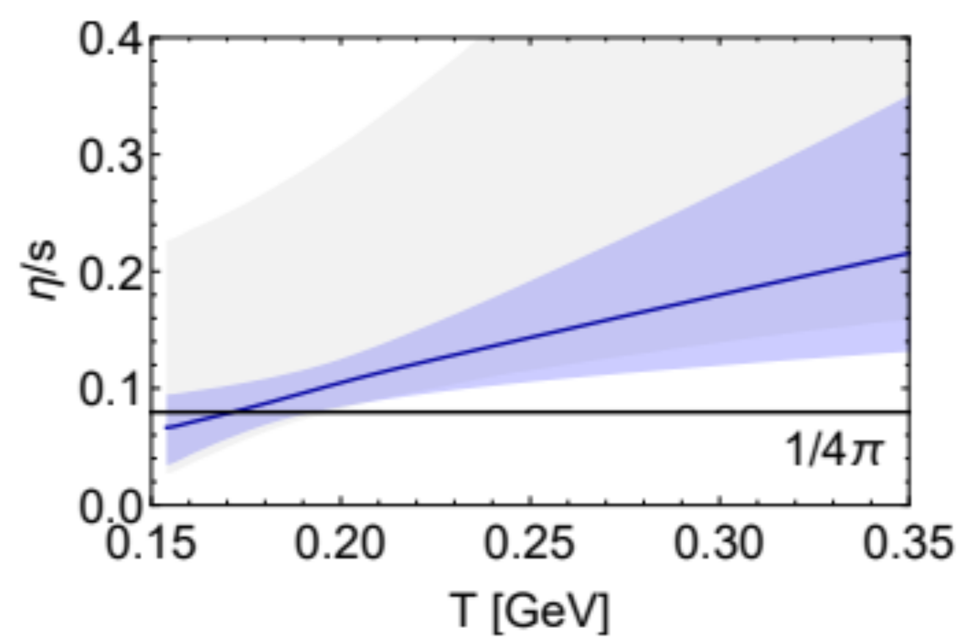
Song, Bass, Heinz, Hirano, Shen, 1011.2783

- **Modern analyses show η/s extracted from data consistent with $1/(4\pi)$ from strongly coupled supersymmetric Yang-Mills theory**

Policastro, Son, Starinets, hep-th/0104066



Bernhard, Moreland, Bass, Nature Physics 15,1113 (2019)



Nijs, van der Schee, Gursoy, Snellings, 2010.15130

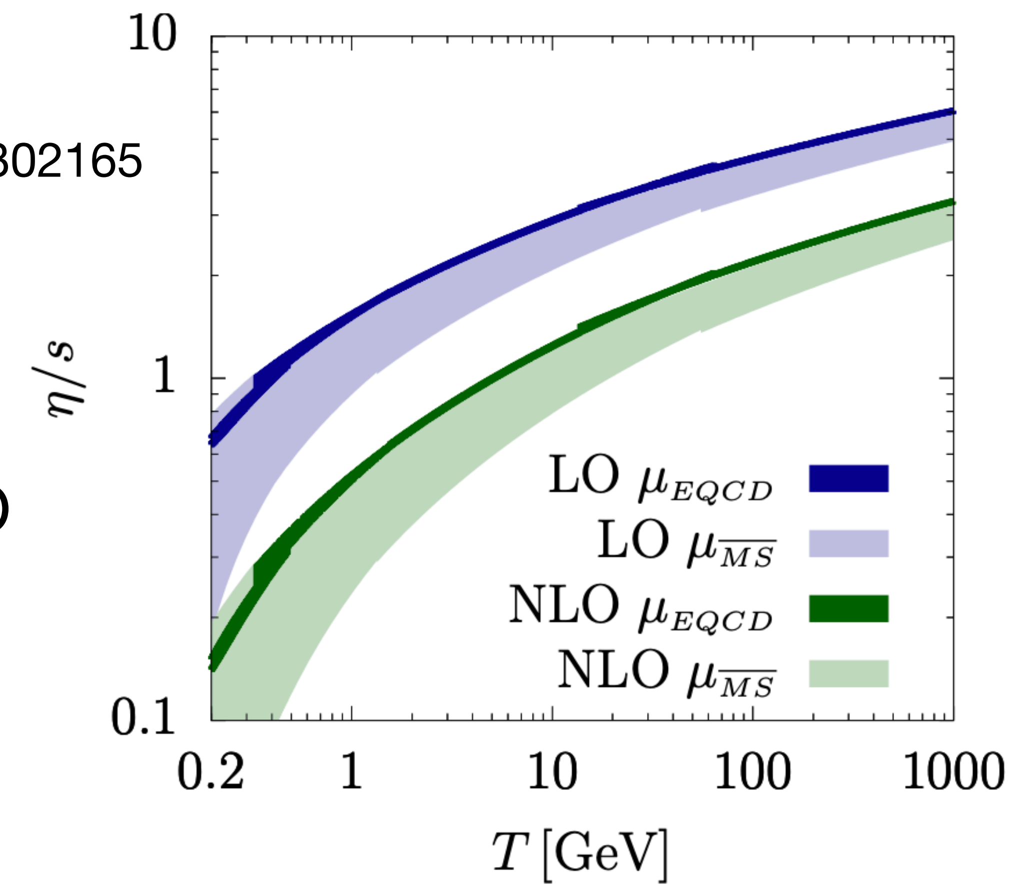
Calculating QCD Shear Viscosity is Challenging

- **Perturbation theory, running coupling**

Jeon, Yaffe, Phys. Rev. D 53, 5799 (1996); Arnold, Moore, Yaffe, hep-ph/0302165

At low T, uncertainty band large

At high T, factor of 2 difference between LO and NLO



Ghiglieri, Moore, Teaney, 1802.09535

- **Euclidean lattice QCD, numerical path integral**

$$G(\tau) = \int \mathrm{d}\boldsymbol{x} \langle T^{xy}(\boldsymbol{x}, i\tau) T^{xy}(0,0)\rangle_T$$

$$G(\tau) = \int \frac{\mathrm{d}\omega}{2\pi} \frac{\rho(\omega)}{\omega} K(\omega,\tau) \qquad K(\omega,\tau) = \frac{\omega \cosh(\omega(\tau - 1/(2T)))}{\sinh(\omega/(2T))}$$

$$\eta = \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

Problems: (1) ill-defined inverse process, different $\rho(\omega)$ can give same $G(\tau)$
(2) Insensitive to structure of $\rho(\omega)$ at small ω

Moore, 2010.15704

Calculation in Real Time

Shear Viscosity from Linear Response

- **Kubo formula: transport determined by real-time correlation function**

"Tree-level" matching $$\eta = \lim_{\omega \to 0} \frac{\partial}{\partial \omega} G_r^{xy}(\omega)$$

Baier, Romatschke, Son, Starinets, Stephanov, 0712.2451

- **Retarded Green's function of T^{xy}**

$$G_r^{xy}(\omega) = \int \mathrm{d}t\, e^{i\omega t} G_r^{xy}(t) \equiv \int \mathrm{d}t\, \mathrm{d}^2 x\, e^{i\omega t} G_r^{xy}(t, \boldsymbol{x})$$

$$G_r^{xy}(t, \boldsymbol{x}) \equiv \theta(t) \mathrm{Tr}\Big([T^{xy}(t, \boldsymbol{x}), T^{xy}(0, \boldsymbol{0})] \rho_T \Big)$$

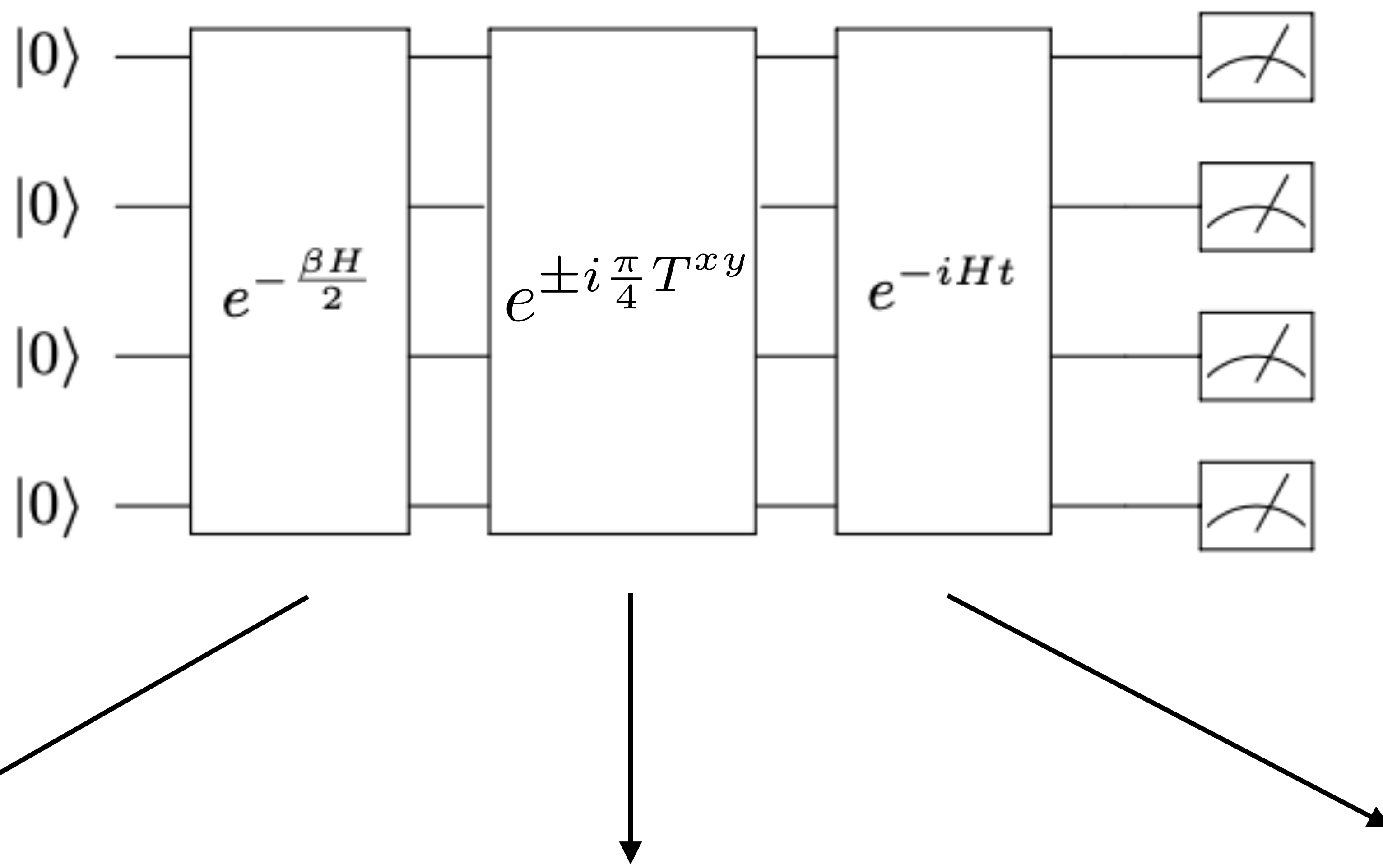
$$T^{\mu\nu} = -\frac{1}{g^2} F^{a\mu\rho} F^{a\nu}{}_{\rho} + \frac{1}{4g^2} \eta^{\mu\nu} F^{a\rho\sigma} F^a_{\rho\sigma}$$

$$\rho_T = \frac{1}{Z} e^{-\beta H}$$

Quantum Algorithm

A Quantum Computing Algorithm

- **An overview**



Measurement and analysis

Thermal state preparation using quantum imaginary time propagation (QITP)

Evolution for commutator if A is a Pauli string

Real-time evolution using Trotterization

Turro, Roggero, Amitrano, Luchi, Wendt, DuBois, Quaglioni, Pederiva, 2102.12260

$$[A, B] = -i\left(e^{-i\frac{\pi}{4}A} B e^{i\frac{\pi}{4}A} - e^{i\frac{\pi}{4}A} B e^{-i\frac{\pi}{4}A}\right)$$

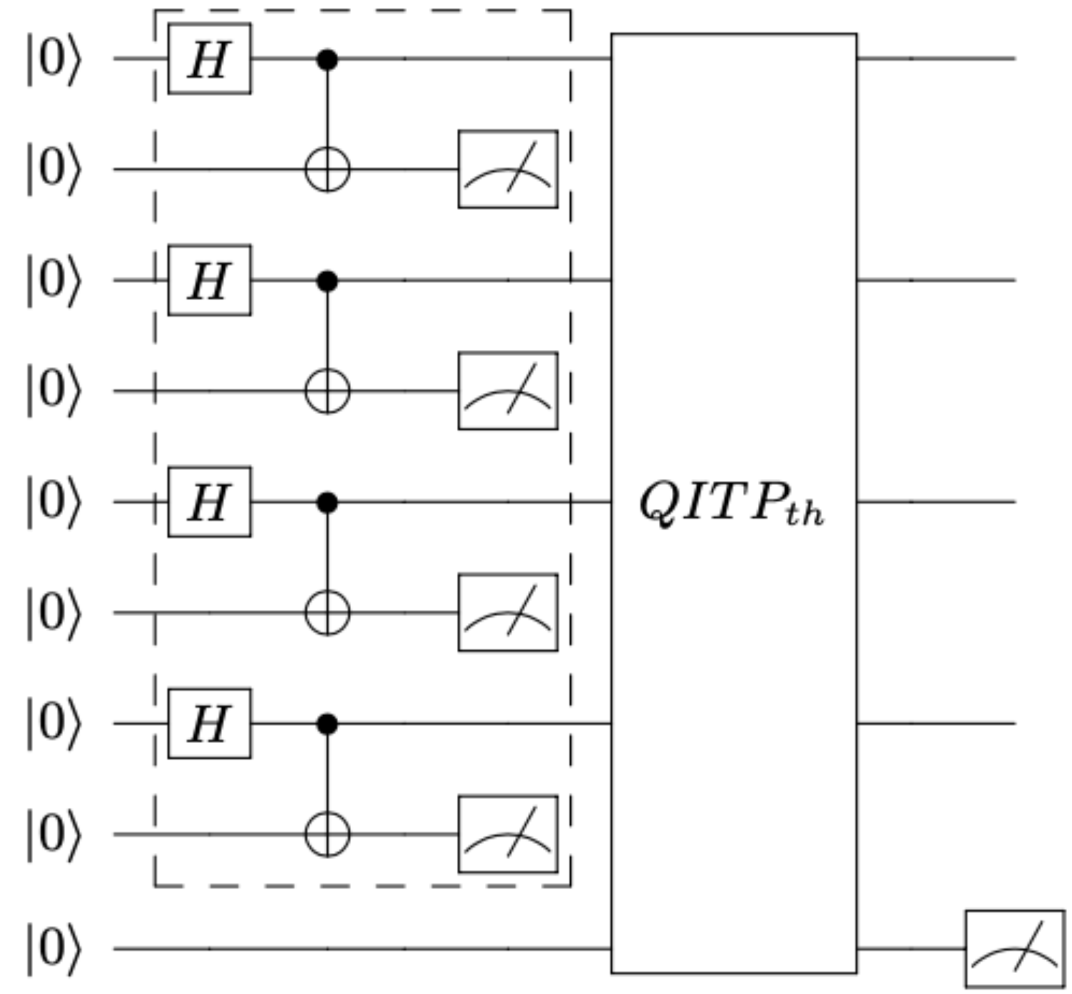
Turro, 2306.16580

Thermal State Preparation

- **Initialization: n_s system qubits + $(n_s + 1)$ ancillas**

Hadamard + CNOT + measurements give maximally mixed state

$$\rho_s = \frac{1}{2^{n_s}} 1_{2^{n_s} \times 2^{n_s}}$$



- **Quantum imaginary time propagation**

$$QITP_{th} = \begin{pmatrix} \sqrt{p}\, e^{-\tau(H-E_T)} & \sqrt{1-p\, e^{-2\tau(H-E_T)}} \\ -\sqrt{1-p\, e^{-2\tau(H-E_T)}} & \sqrt{p}\, e^{-\tau(H-E_T)} \end{pmatrix}$$

- **Measure the ancilla and if $|0\rangle$ returned**

$$\rho_T = \frac{1}{2^{n_s} p_s} e^{-\beta(H-E_T)} = \frac{1}{Z} e^{-\beta H}$$

$$p = 1$$

$$\tau = \frac{\beta}{2}$$

Quantum Computing of Retarded Green's Function

- **Commutator from a unitary circuit (A is a Pauli string)**

$$[A, B] = -i\left(e^{-i\frac{\pi}{4}A}Be^{i\frac{\pi}{4}A} - e^{i\frac{\pi}{4}A}Be^{-i\frac{\pi}{4}A}\right)$$

- **Run different circuits to obtain retarded Green's function of T^{xy}**

$$[T^{xy}_{\rm sum}(t), T^{xy}_{ij}(0)] = [T^{xy}_{\rm sum}(t), \sum_\alpha \Sigma_\alpha]$$

$$[T^{xy}_{\rm sum}(t), \Sigma_\alpha] = ie^{-i\frac{\pi}{4}\Sigma_\alpha}e^{iHt}T^{xy}_{\rm sum}e^{-iHt}e^{i\frac{\pi}{4}\Sigma_\alpha} - ie^{i\frac{\pi}{4}\Sigma_\alpha}e^{iHt}T^{xy}_{\rm sum}e^{-iHt}e^{-i\frac{\pi}{4}\Sigma_\alpha}$$

- **Measure in computational basis and post-processing**

$$\mathrm{Tr}([T^{xy}_{\rm sum}(t), \Sigma_\alpha]\rho_T) = i\sum_b \langle b|T^{xy}_{\rm sum}(0)|b\rangle[P^+_\alpha(b) - P^-_\alpha(b)]$$

b → Basis state

Application to 2+1D SU(2) Pure Gauge Theory

Kogut-Susskind Hamiltonian

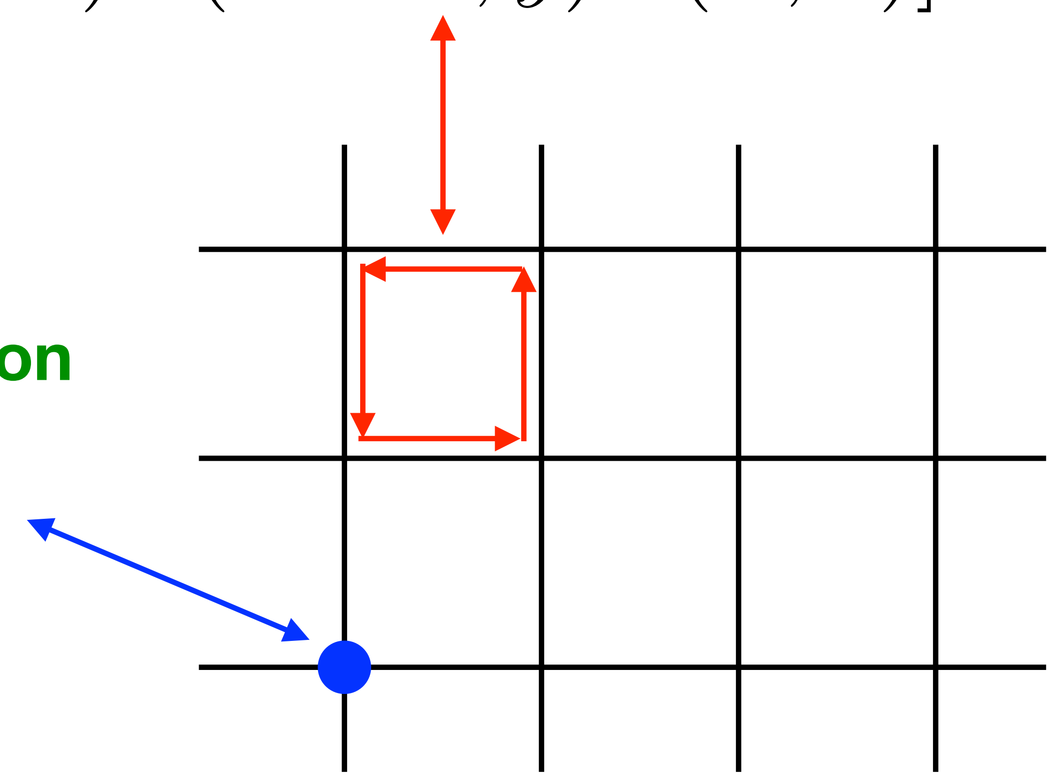
- **On spatial lattice**

$$H = \frac{g^2}{2} \sum_{\text{links}} (E_i^a)^2 - \frac{2}{a^2 g^2} \sum_{\text{plaquettes}} \Box(\boldsymbol{n})$$

- **Plaquette term consists of four gauge links**

$$\Box(\boldsymbol{n}) = \text{Tr}[U^\dagger(\boldsymbol{n}, \hat{y}) U^\dagger(\boldsymbol{n} + \hat{y}, \hat{x}) U(\boldsymbol{n} + \hat{x}, \hat{y}) U(\boldsymbol{n}, \hat{x})]$$

$$U(\boldsymbol{n}, \hat{i}) = e^{iaA_i(\boldsymbol{n})}$$



- **Electric fields generate gauge transformation**

$$[E_i^a, U(\boldsymbol{n}, \hat{j})] = -\delta_{ij} T^a U(\boldsymbol{n}, \hat{j})$$

$$[E_i^a, E_i^b] = i f^{abc} E_i^c$$

$$\sum_{i \in \text{vertex}} E_i^a = 0$$ Gauss's law

Byrnes, Yamamoto, quant-ph/0510027

Electric Basis and Gauss's Law

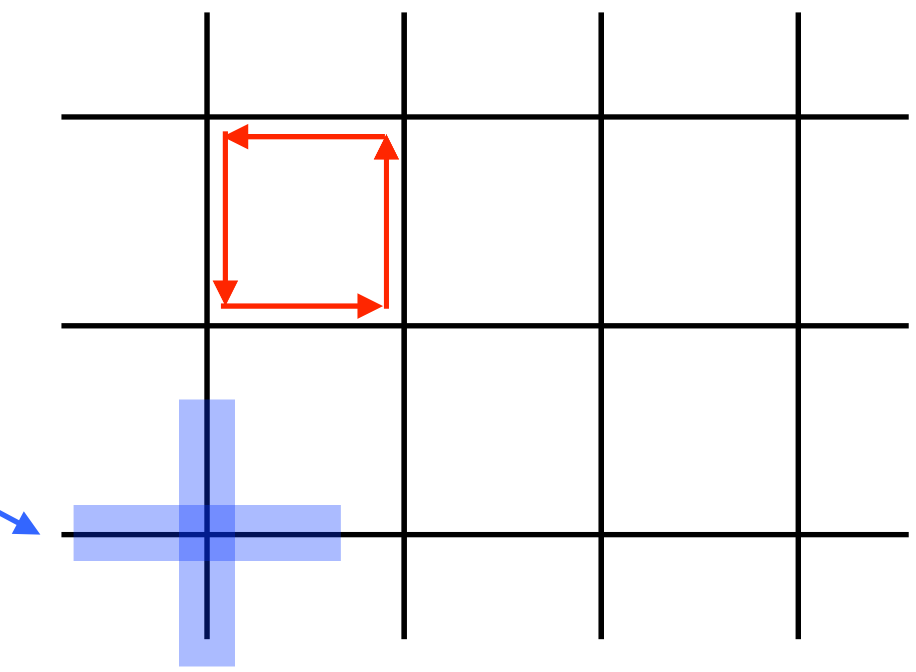
- **Electric basis on links:** $|j\, m_L\, m_R\rangle$ $\qquad |j\ m_L\rangle$ ——— $|j\ m_R\rangle$

$$E^2|j\, m_L\, m_R\rangle = j(j+1)|j\, m_L\, m_R\rangle$$

Similar to angular momentum quantum numbers

- **Only gauge invariant states are physical**

Impose Gauss's law: physical states transform as **SU(2) singlet** at each vertex

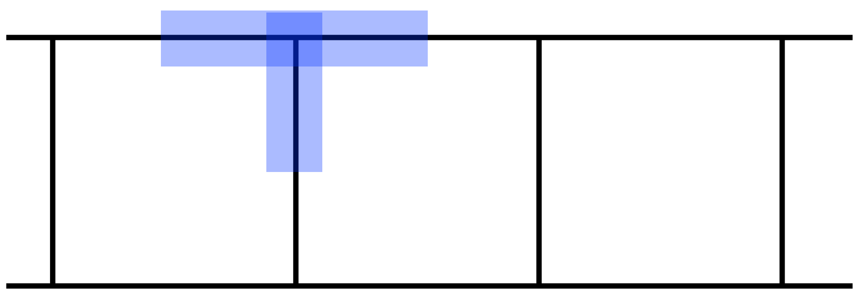


E.g. two links with $j = \dfrac{1}{2}$

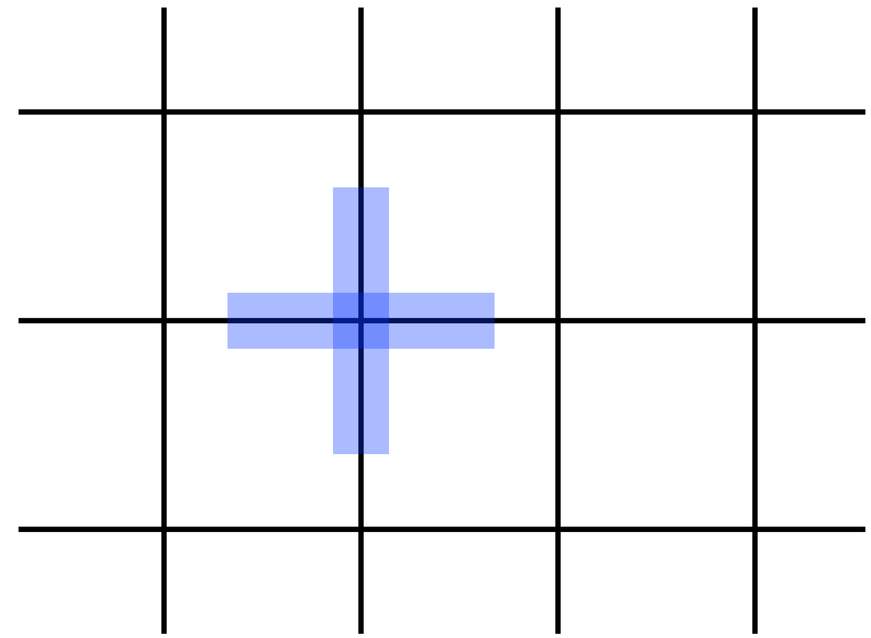
$$\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle \to |0,0\rangle$$

Honeycomb Lattice

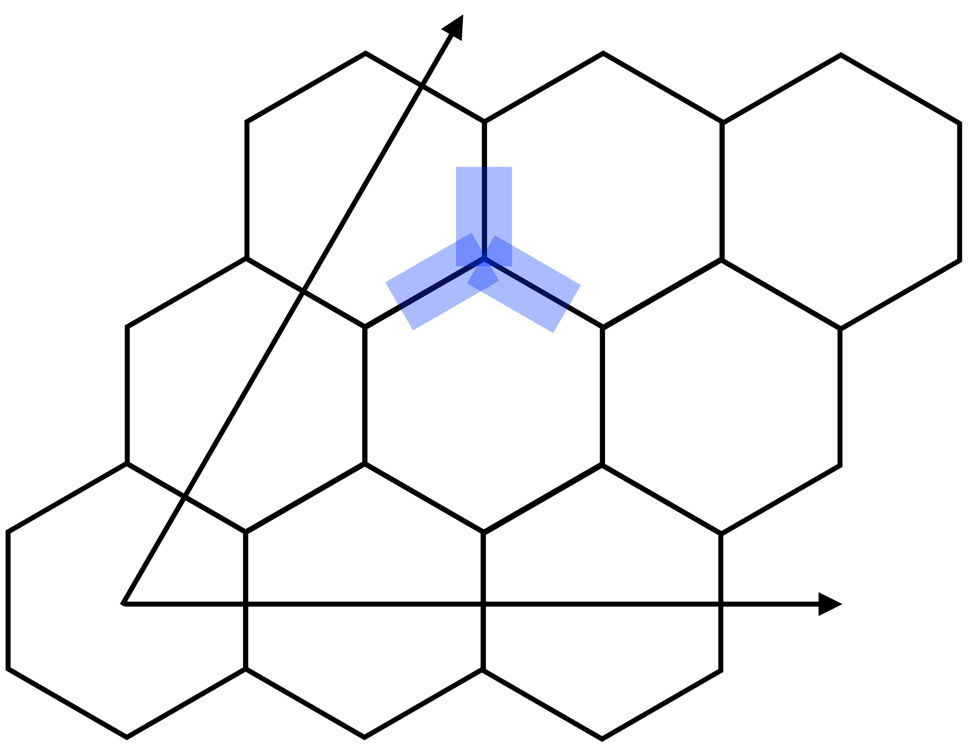
- **Problem on square lattice:** each vertex has four links —> singlet is **not uniquely** defined by four j values



Klco, Stryker, Savage, 1908.06935



- **Use honeycomb lattice**



Müller, XY, 2307.00045

$$H_{\text{el}} = \frac{g^2}{2}\frac{3\sqrt{3}}{2}\sum_{\text{links}} E_i^a E_i^a$$

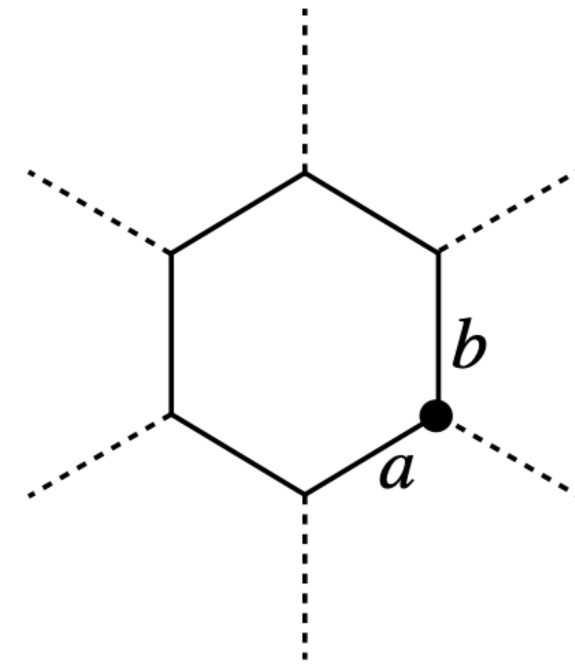
$$H_{\text{mag}} = -\frac{4\sqrt{3}}{9a^2g^2}\sum_{\text{plaqs}} \hexagon$$

Matrix Elements of Hamiltonian and T^{xy}

- **Plaquette matrix element in electric basis**

$$\langle \{J\} | \hexagon | \{j\} \rangle \equiv \langle \{J\} | \prod_{V=1}^{6} M_V | \{j\} \rangle$$

$$= \prod_{V=1}^{6} (-1)^{j_a + J_b + j_x} \sqrt{(2J_a+1)(2j_b+1)} \begin{Bmatrix} j_x & j_a & j_b \\ \frac{1}{2} & J_b & J_a \end{Bmatrix}$$



Klco, Stryker, Savage, 1908.06935

Zache, González-Cuadra, Zoller, 2304.02527

Hayata, Hidaka, 2305.05950

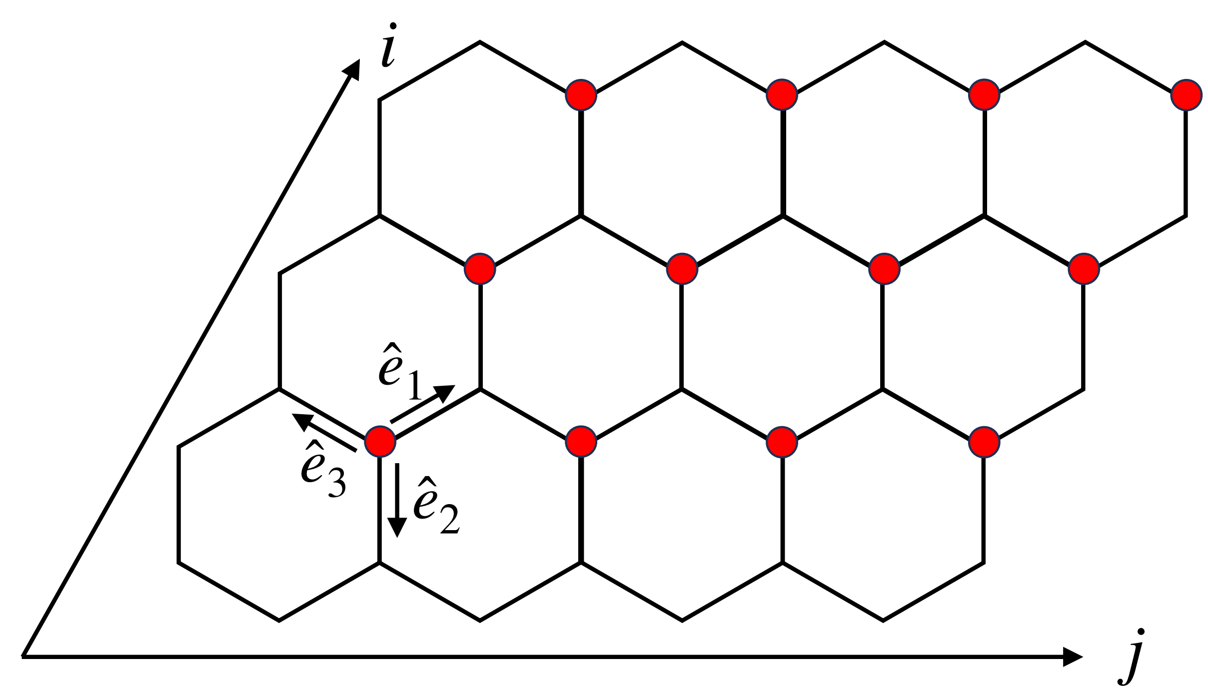
Each vertex (V) has two internal links (a, b) and one external (x)

- **T^{xy} operator** $T^{xy} = -\dfrac{g^2}{a^2} E_x^a E_y^a$

$$E_1^a + E_2^a + E_3^a = 0$$

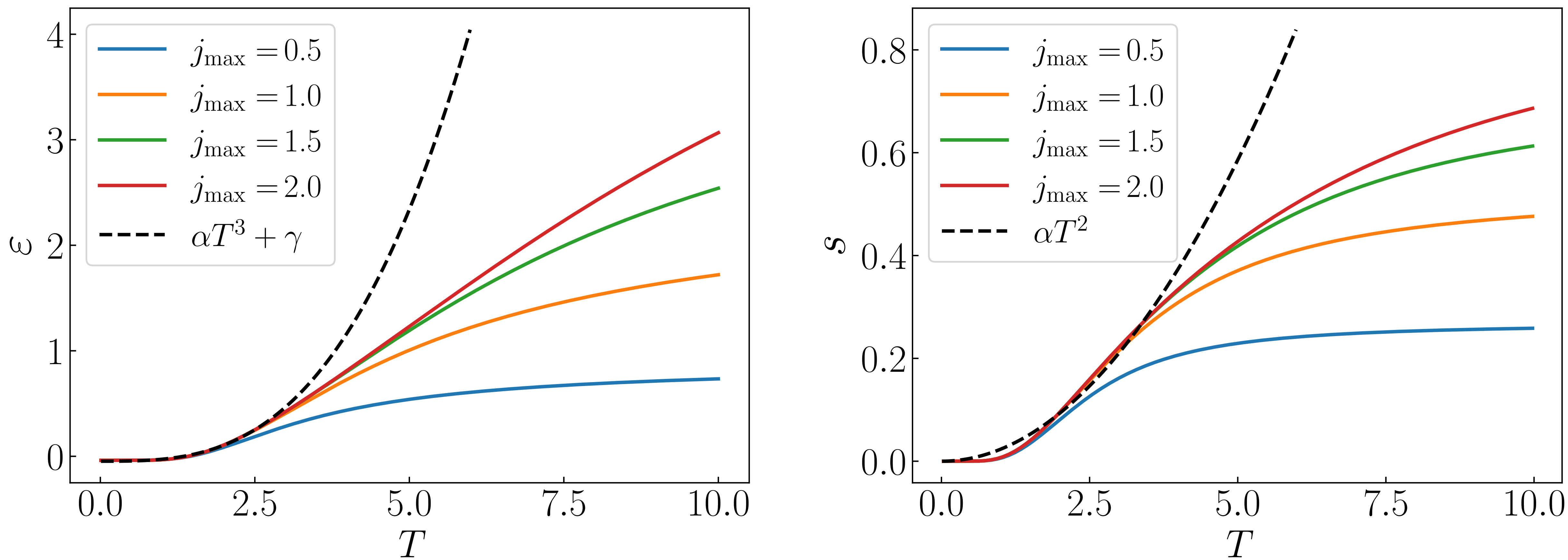
$$\downarrow$$

$$T^{xy} = -\frac{g^2}{\sqrt{3}a^2}\left((E_1^a)^2 - (E_3^a)^2\right)$$



$j_{\max}$ Cutoff Effect

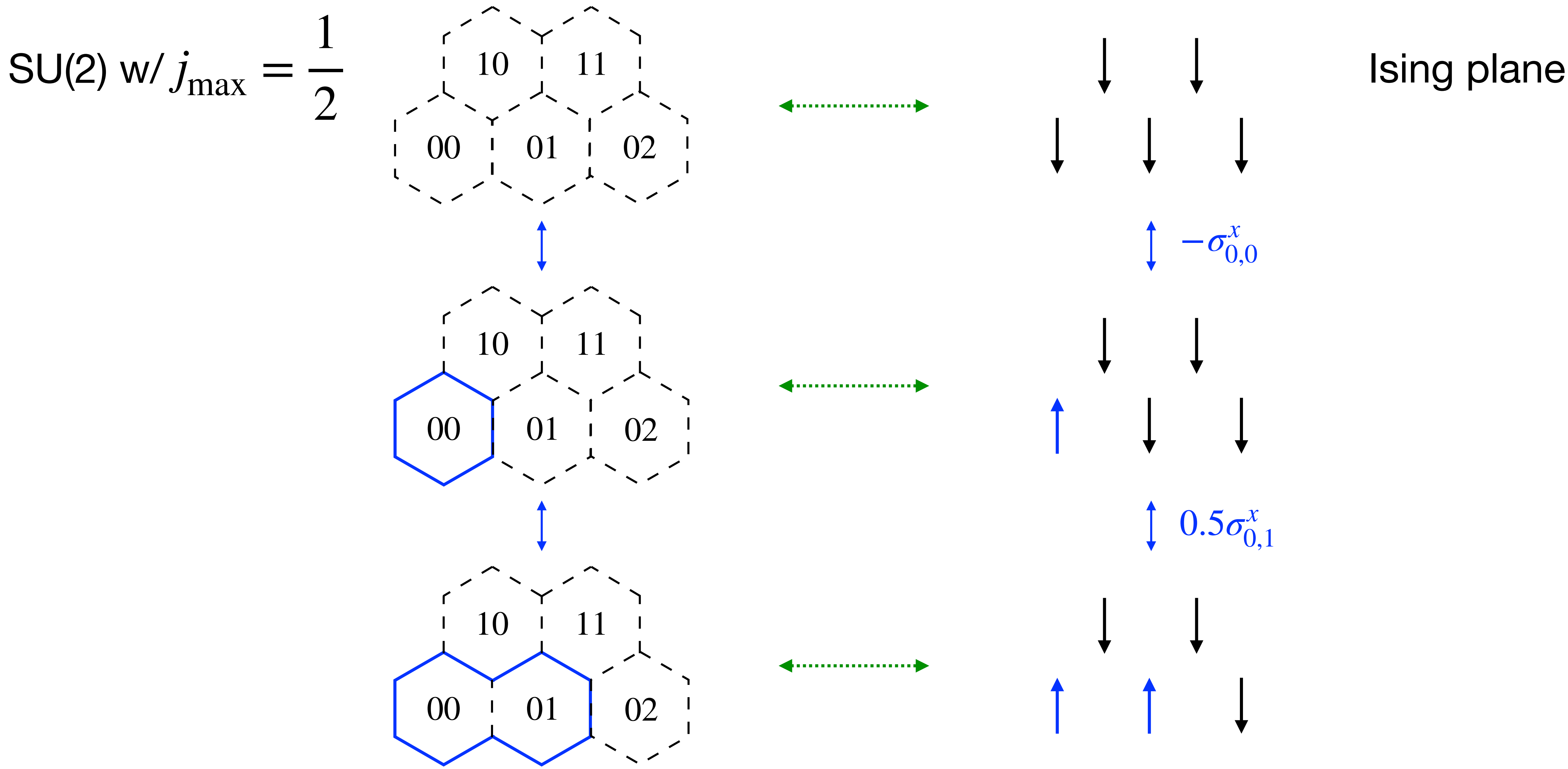
- **Energy and entropy densities on 2×2 lattice with $ag^2 = 1$**



- **To describe states up to energy E with error ϵ, we need at most**

$$j_{\max} = \frac{4N_l\widetilde{E}}{3\sqrt{3}g^2\epsilon} \qquad \widetilde{E} = E + \frac{16\sqrt{3}}{9g^2a^2}N_p$$

Simplify Hamiltonian with $j_{\max} = 1/2$



$$aH = h_+ \sum_{(i,j)} \Pi^+_{i,j} - h_{++} \sum_{(i,j)} \Pi^+_{i,j} \left(\Pi^+_{i+1,j} + \Pi^+_{i,j+1} + \Pi^+_{i+1,j-1} \right) + h_x \sum_{(i,j)} (-0.5)^{c_{i,j}} \sigma^x_{i,j}$$

$$\Pi^+_{i,j} = (1 + \sigma^z_{i,j})/2 \qquad h_+ = \frac{27\sqrt{3}}{8} a g^2 \,, \; h_{++} = \frac{9\sqrt{3}}{8} a g^2 \,, \; h_x = \frac{4\sqrt{3}}{9 a g^2}$$

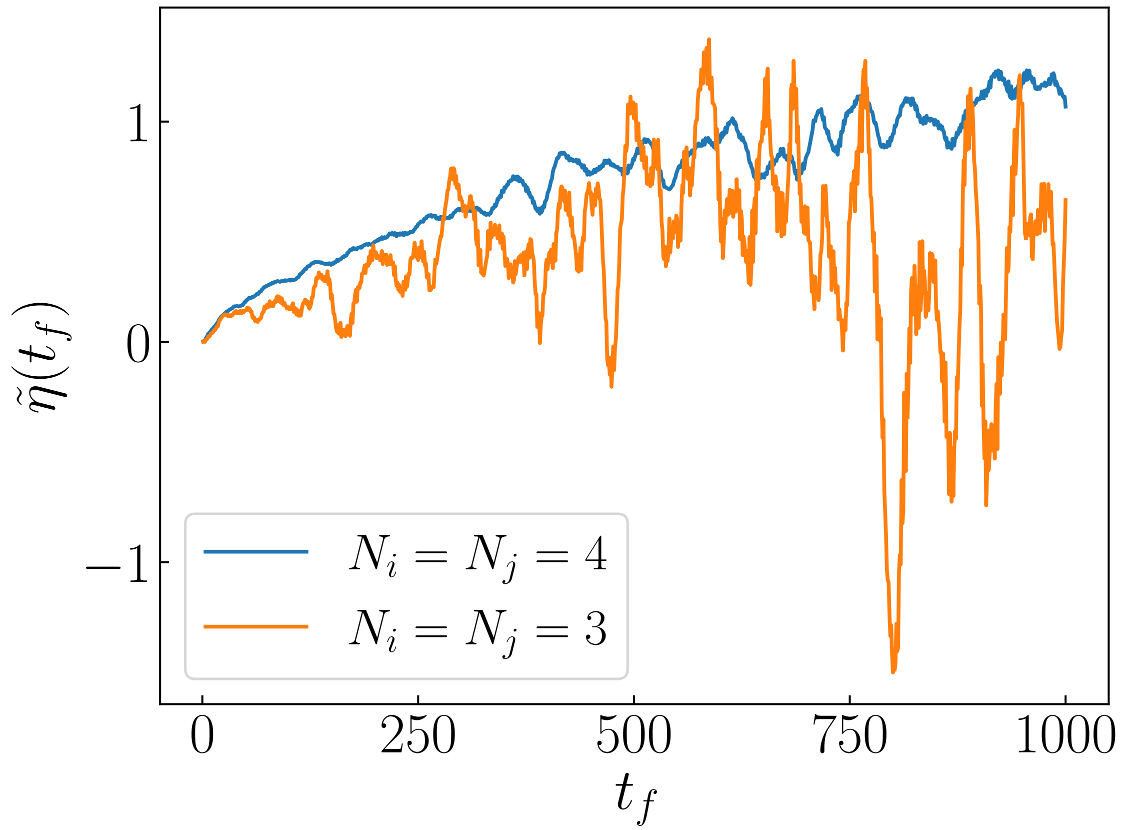
XY, 2303.14264; Müller, XY, 2307.00045

Classical Results

Results at Fixed Coupling for $j_{\max} = 1/2$ Model

- **Finite size effect**

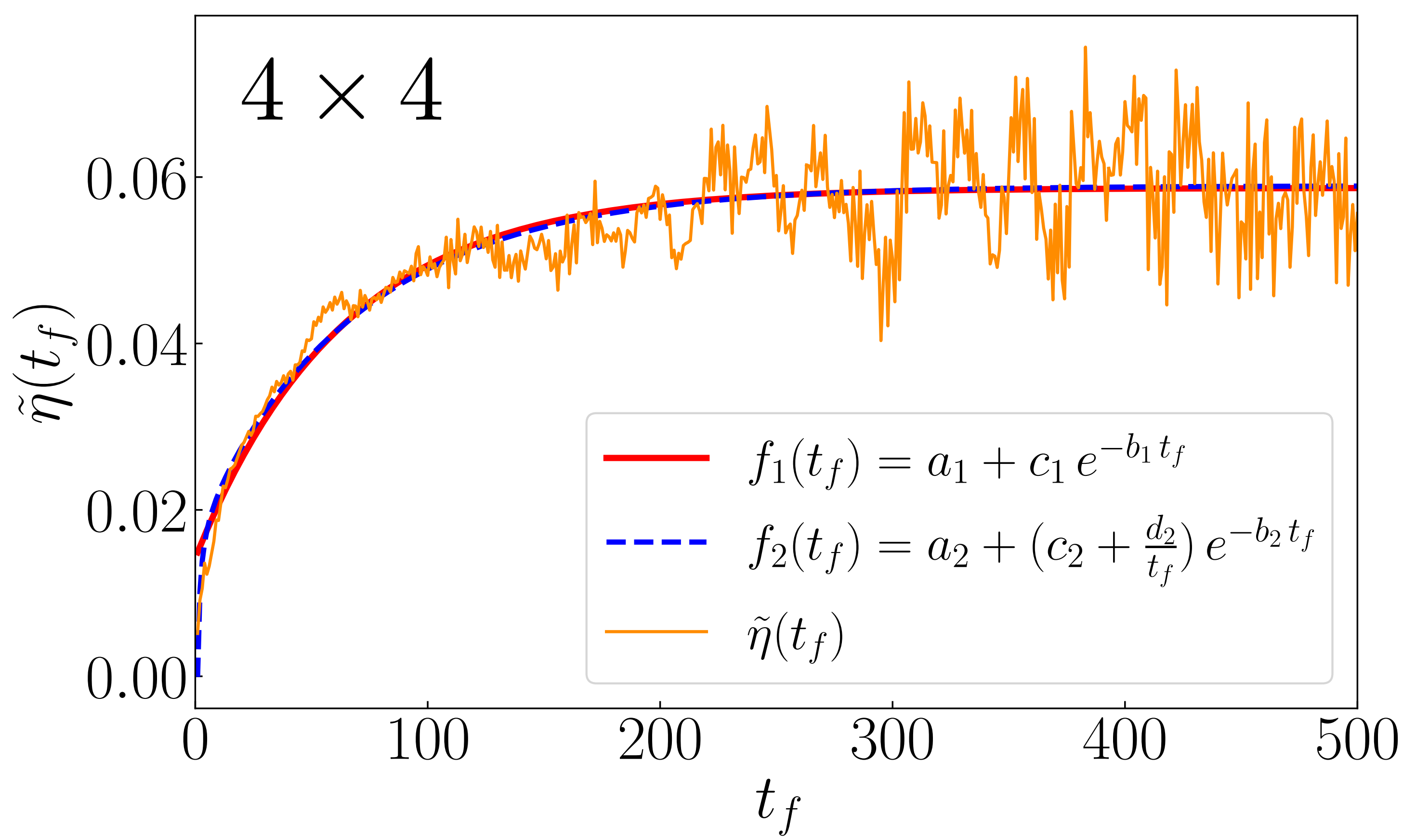
$$|\widetilde{T}^{xy}_{nm}|^2 \Big(\frac{\sin((E_n - E_m)t_f)}{(E_n - E_m)^2} - \frac{t_f \cos((E_n - E_m)t_f)}{E_n - E_m} \Big)$$



$\beta = 0.3a$

$ag^2 = 1$

- **Fit plateau value**



$\beta = 0.2a$

$ag^2 = 0.6$

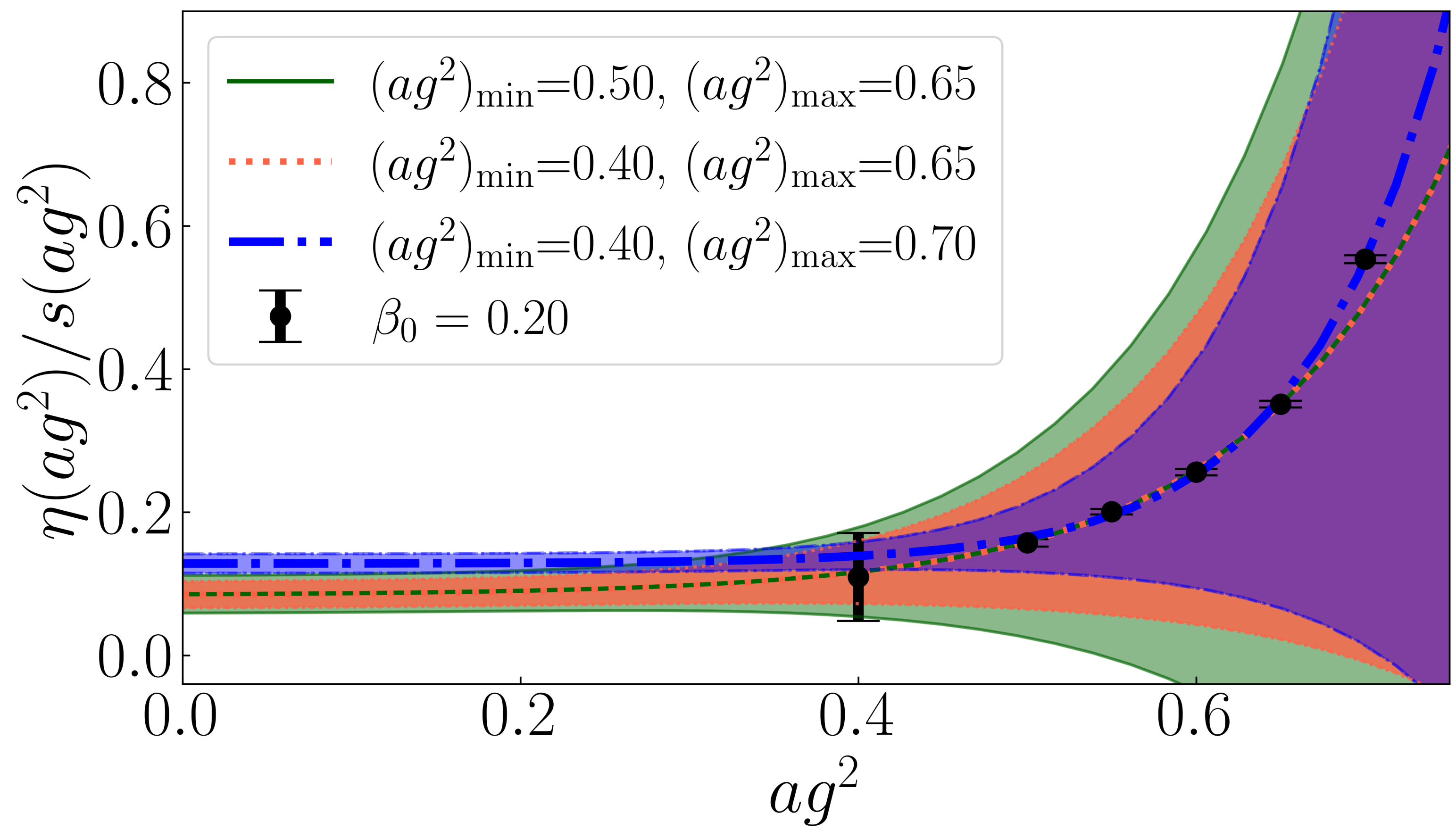
Running Coupling and "Continuum" Limit

- **Renormalization of coupling**

$$\frac{\mathrm{d}\ln(ag^2)}{\mathrm{d}\ln a} = 1$$

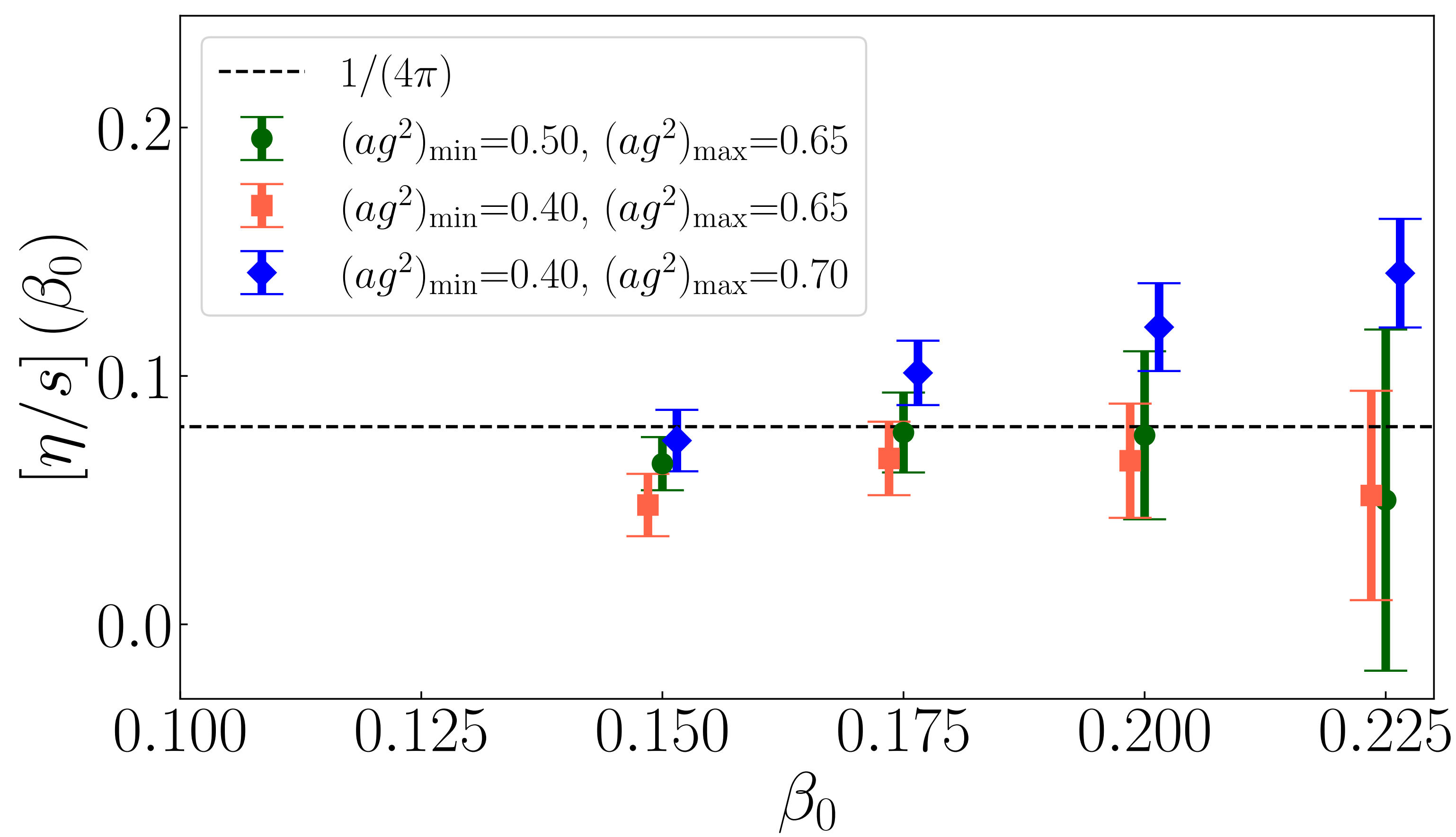
Romatschke, 1910.09550

$$f(ag^2) = c_0 + c_1 e^{c_2 ag^2}$$



- **Temperature dependence for truncated lattice model**
  $4 \times 4,\ j_{\rm max} = 1/2$

β_0 in lattice unit is the temperature when $ag^2 = 1$



How to Improve the Results

- Hamiltonian lattice formulation allows us to evaluate real-time correlation for shear viscosity extraction

- Physical limit: (1) $a \to 0$ means $ag^2 \to 0$, requires $j_{\rm max} \to \infty$
  (2) lattice size $\to \infty$
  (3) Operator renormalization

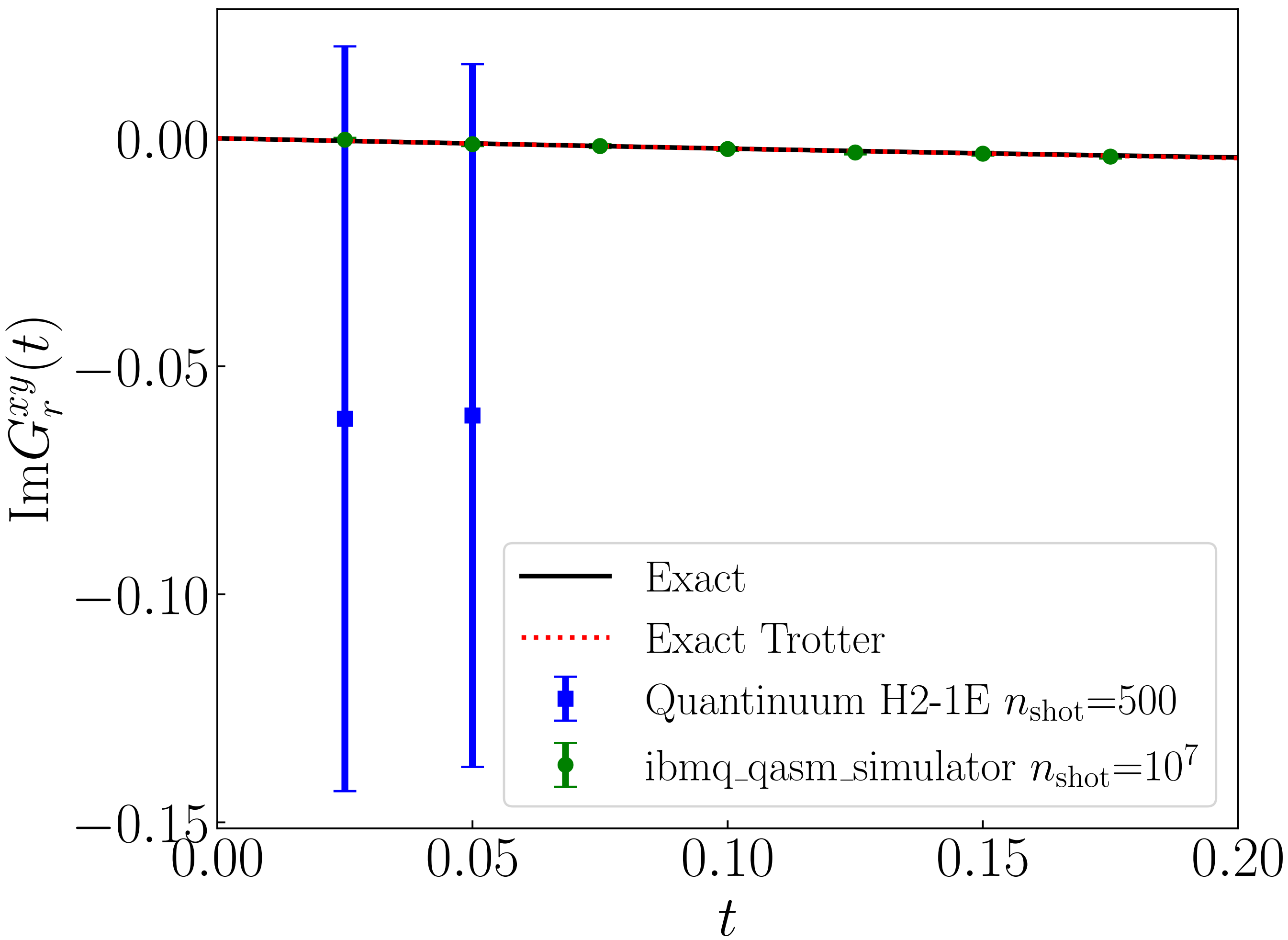
- (1) and (2) are challenging: 4×4 lattice w/ $j_{\rm max} = 1/2$ has 65536 states
  3×3 lattice w/ $j_{\rm max} = 1$ has 519233 states

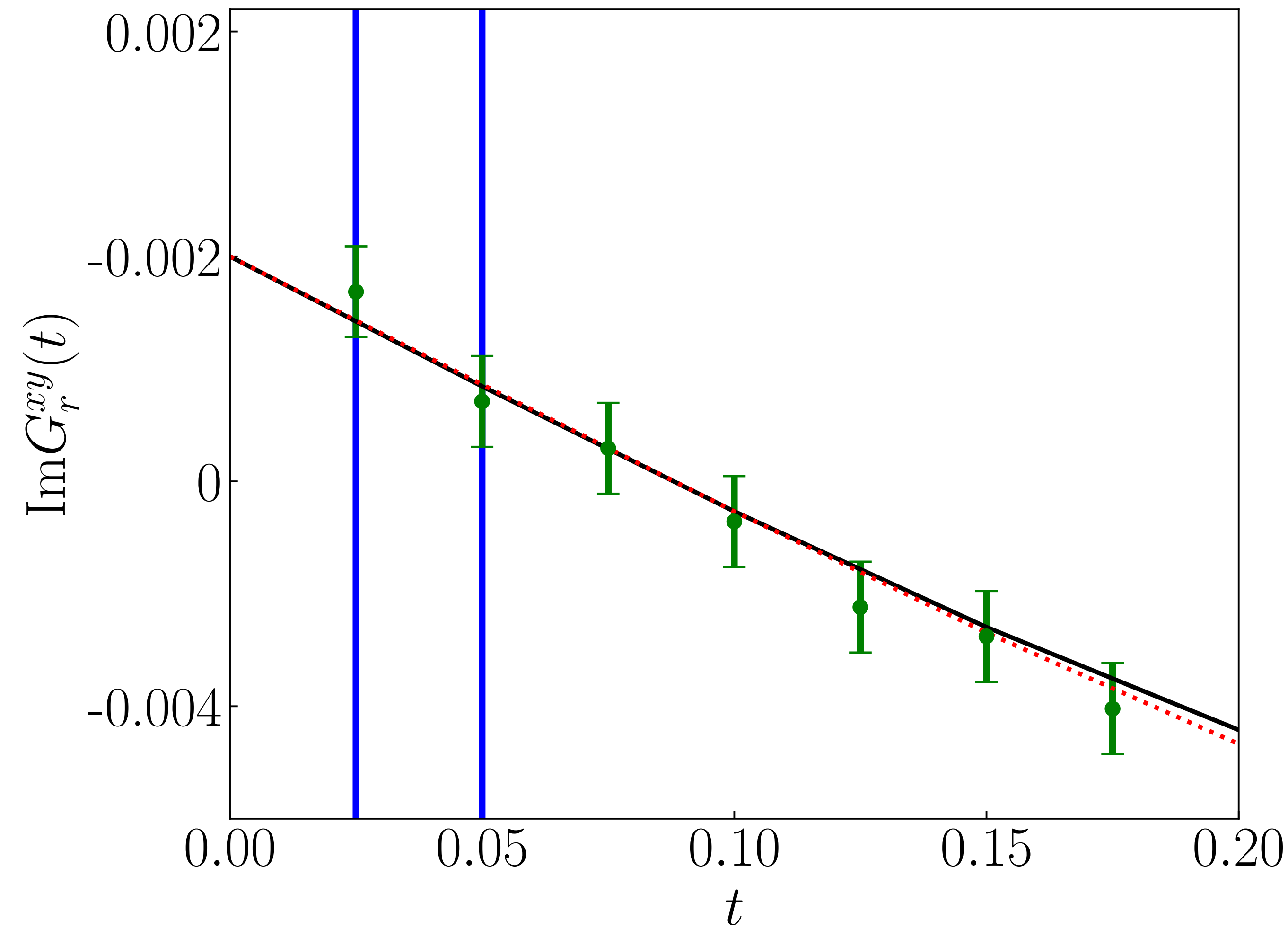
- Exact diagonalization cannot take us too far —> quantum computing

Quantum Simulator Results

Preliminary Results on Small Lattice

- **Quantum simulator results for 2×2 lattice with** $j_{\max} = 1/2,\ ag^2 = 1,\ \beta = 0.15,\ \Delta t = 0.025$





Many shots are needed!

$$n_{\mathrm{shot}} \simeq \frac{4\, d_T^2}{\epsilon^2 [G_r^{xy}(t)]^2} \sim \frac{4 \times 10^6\, d_T^2}{\epsilon^2}$$

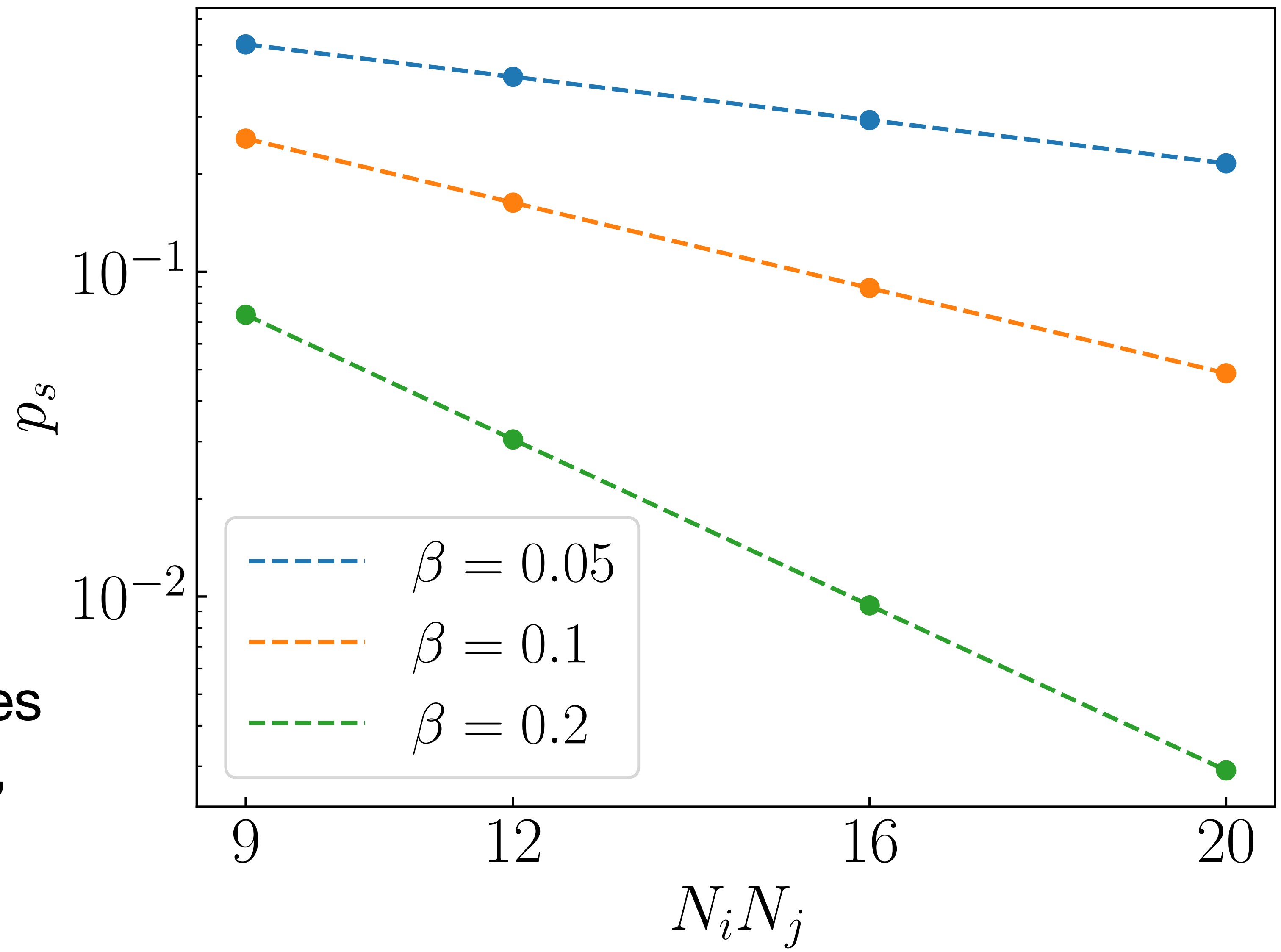
$$|\langle b | T^{xy}_{\mathrm{sum}}(0) | b \rangle| \leq d_T$$

Thermal State Preparation Efficiency

- **Success probability**

Fixed $j_{\max} = \frac{1}{2}$, $ag^2 = 1$

"Glueball mass":
$E_1 - E_0 = 6.2$



Success probability decreases exponentially w/ system size, but for high temperature, coefficient is small

Conclusions

- Shear viscosity: interesting physical quantity but hard to compute in QCD
- Real-time Hamiltonian lattice approach:
  - Classical computing: exact diagonalization up to 4×4 lattice with $j_{\text{max}} = 1/2$; model results show consistency with $\eta/s = 1/(4\pi)$ in naive “continuum” limit
  - A quantum computing algorithm
- Future goal: **approach the physical limit**

Backup: Shear Viscosity from Linear Response

- **Express in terms of eigenstates and eigenenergies** $H|n\rangle = E_n|n\rangle$

$$\eta = \lim_{t_f \to \infty} \tilde{\eta}(t_f)$$

$$\tilde{\eta}(t_f) \equiv -\int_0^{t_f} t\,\mathrm{d}t\,\mathrm{Im}G_r^{xy}(t)$$

$$= -\frac{2}{Z\mathcal{A}} \sum_n \sum_{m \neq n} |\langle n|\widetilde{T}^{xy}|m\rangle|^2 e^{-\beta E_n} f(t_f)$$

$$f(t_f) \equiv \frac{\sin((E_n - E_m)t_f)}{(E_n - E_m)^2} - \frac{t_f \cos((E_n - E_m)t_f)}{E_n - E_m}$$

- **Assume translational invariance**

$$\widetilde{T}^{xy}(t) = \int \mathrm{d}^2x\, T^{xy}(t, \boldsymbol{x})$$

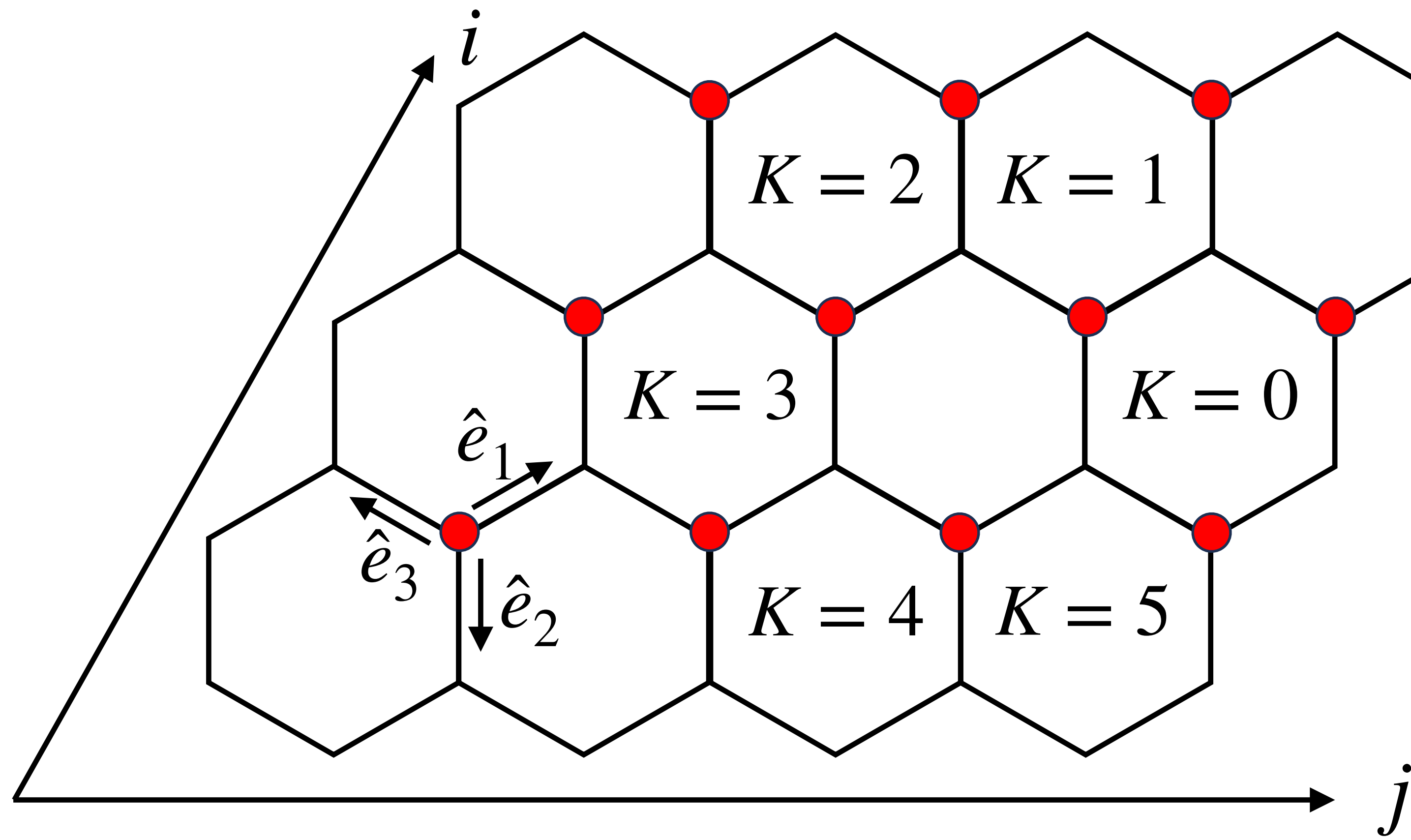
Backup: Quantum Circuit Gives G_r^{xy}

- **What the circuit does:** $$\rho_\alpha^\pm(t) = \frac{1}{Z} U_t e^{\pm i\frac{\pi}{4}\Sigma_\alpha} e^{-\beta H} e^{\mp i\frac{\pi}{4}\Sigma_\alpha} U_t^\dagger$$
- **What the measurement does:**

$$\sum_b \langle b|T^{xy}_{\rm sum}(0)|b\rangle P_\alpha^\pm(b) = {\rm Tr}[T^{xy}_{\rm sum}(0)\rho_\alpha^\pm(t)]$$
$$= \frac{1}{Z}{\rm Tr}[e^{\mp i\frac{\pi}{4}\Sigma_\alpha} U_t^\dagger T^{xy}_{\rm sum}(0) U_t e^{\pm i\frac{\pi}{4}\Sigma_\alpha} e^{-\beta H}]$$
$$= \frac{1}{Z}{\rm Tr}[e^{\mp i\frac{\pi}{4}\Sigma_\alpha} T^{xy}_{\rm sum}(t) e^{\pm i\frac{\pi}{4}\Sigma_\alpha} e^{-\beta H}]$$

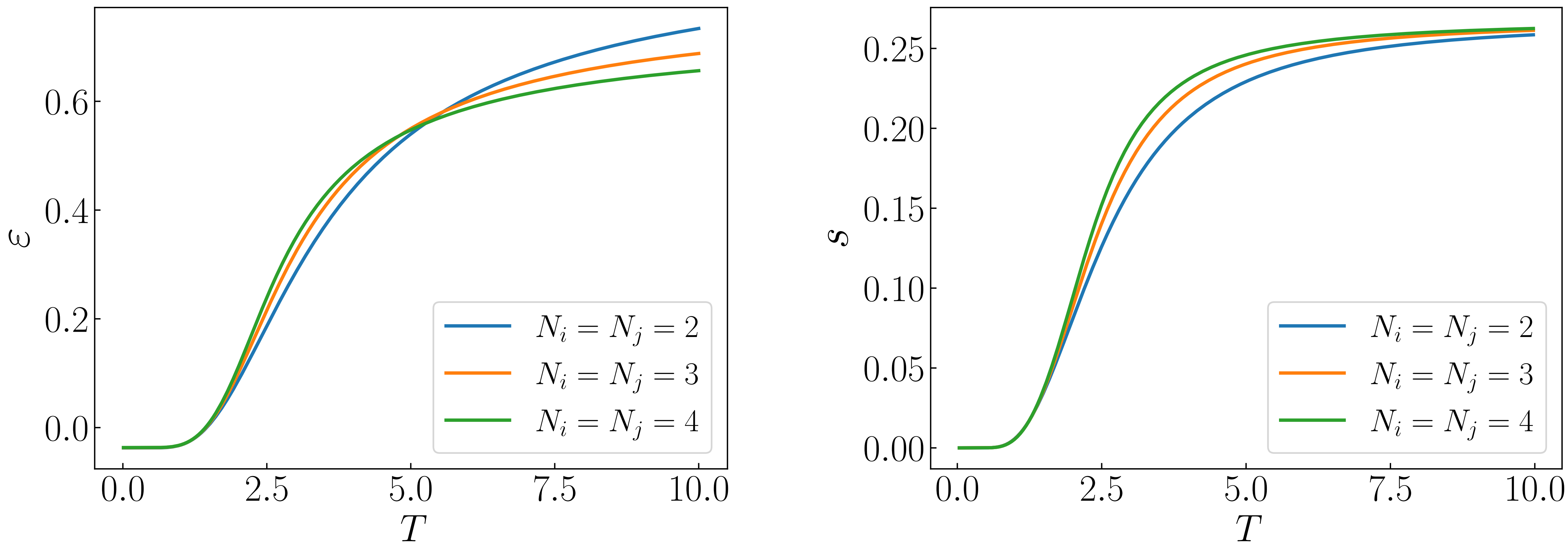
$${\rm Tr}[T^{xy}_{\rm sum}(0)\rho^+(t)] - {\rm Tr}[T^{xy}_{\rm sum}(0)\rho^-(t)]$$
$$= {\rm Tr}([e^{-i\frac{\pi}{4}\Sigma_\alpha} T^{xy}_{\rm sum}(t) e^{i\frac{\pi}{4}\Sigma_\alpha} - e^{i\frac{\pi}{4}\Sigma_\alpha} T^{xy}_{\rm sum}(t) e^{-i\frac{\pi}{4}\Sigma_\alpha}]\rho_T)$$
$$= \frac{-i}{Z}{\rm Tr}([T^{xy}_{\rm sum}(t), \Sigma_\alpha]\, e^{-\beta H})$$

Backup: Magnetic Interaction



$$H^{\mathrm{mag}} = h_x \sum_{(i,j)} \sigma^x_{i,j} \prod_{K=0}^{5} \left[\left(\frac{1}{2} - \frac{i}{2\sqrt{2}} \right) \sigma^z_K \sigma^z_{K+1} + \frac{1}{2} + \frac{i}{2\sqrt{2}} \right]$$

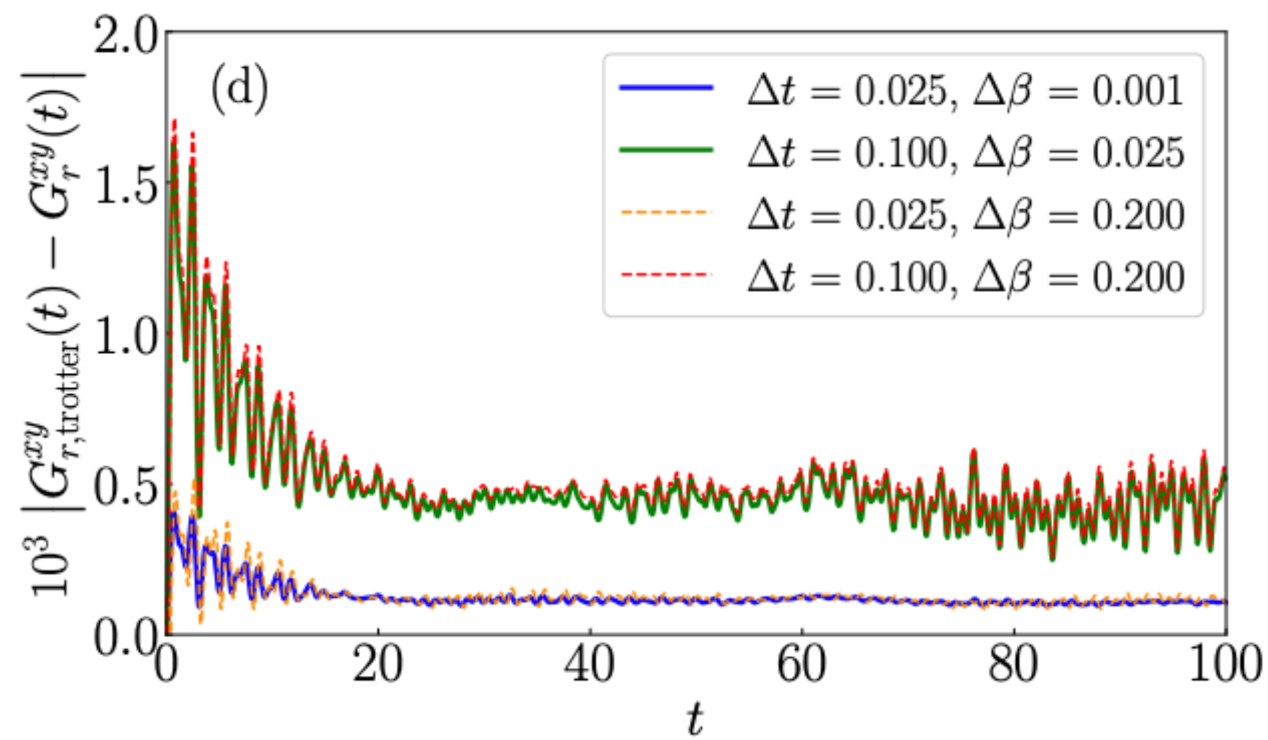
Backup: Volume Dependence of Energy and Entropy Densities



Backup: Systematic Uncertainties

- **Trotter errors in real-time and *QITP***

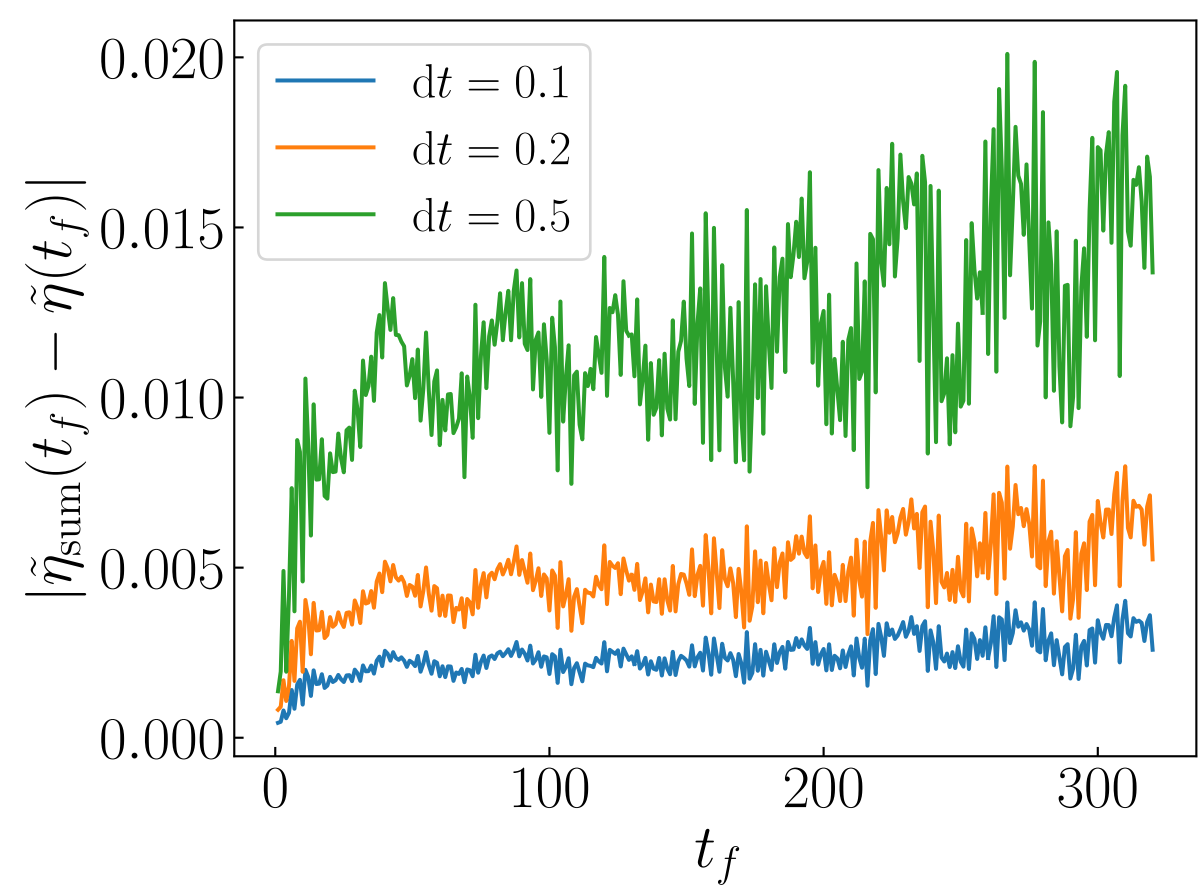
Trotter error in *QITP* is negligible



- **Integration error from Riemann sum**

$$\tilde{\eta}_{\mathrm{sum}}(t_f) \equiv -(\Delta t)^2 \sum_{k=1}^{N_t} k\, \mathrm{Im} G_r^{xy}(k\Delta t)$$

Important to determine how often to do measurements in the circuit



Backup: Spectral Function at Small Frequency

- **Relation between spectral function and off-diagonal matrix elements**

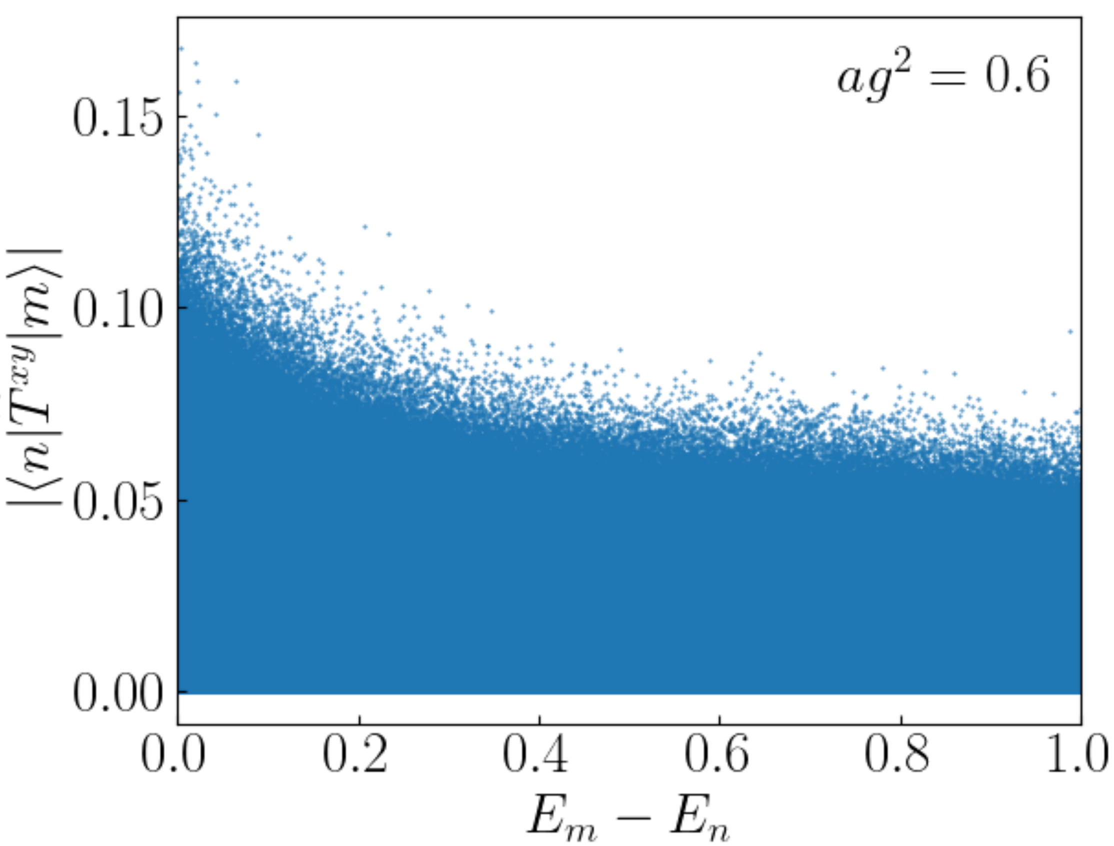
$$\rho^{xy}(\omega) \equiv \frac{1}{\mathcal{A}} \int \mathrm{d}t\, e^{i\omega t} \mathrm{Tr}\big([\widetilde{T}^{xy}(t), \widetilde{T}^{xy}(0)]\rho_T\big)$$

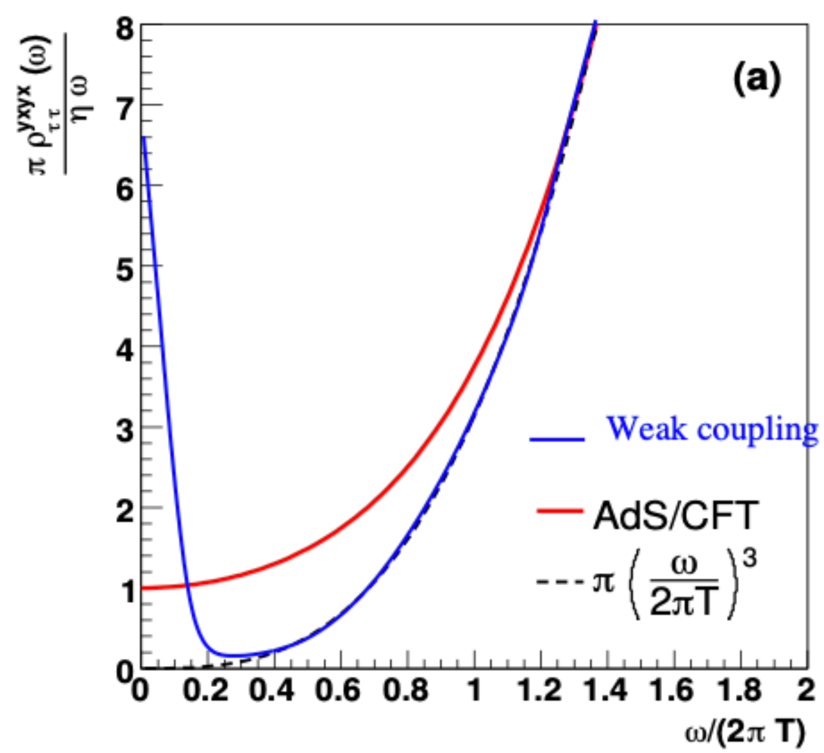
$$= \frac{1}{\mathcal{A}Z} \sum_n \sum_m 2\pi\delta(\omega + E_n - E_m)|\langle n|\widetilde{T}^{xy}|m\rangle|^2 (e^{-\beta E_n} - e^{-\beta E_m})$$

$$\downarrow$$

$$e^{-\beta E_n}[\beta\omega + O(\omega^2)]$$

- $\dfrac{\rho^{xy}(\omega)}{\omega}$ **exhibits peak structure**





Moore, 2010.15704